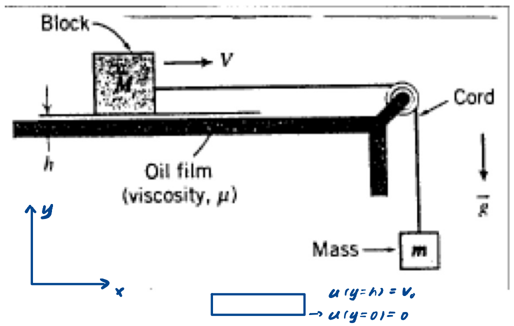
Tutorial 1

1. A cube with mass M slides on a thin film of oil with viscosity μ. The width of film is h and is constant, and the area of the face of the cube which is in contact with the oil is A. When a cube of mass m is released, tension results in the rope, which pulls with it the cube with mass M. Neglecting air resistance and friction in the pulley.



   A. Express the viscous force which acts on the cube of mass M.

   B. Write a differential equation which describes the velocity of M as a function of time.

   C. Solve that differential equation and identify the velocity at infinite time of M.

A. Shear stress $\tau_{yx} = \mu \dot{\gamma} = \mu \frac{du}{dy}$

Viscous force $F_v = \tau_{yx} \cdot \vec{A}$

No Slip condition

$$\begin{cases} u(y=0) = 0 \\ u(y=h) = V \end{cases} \Rightarrow u(y) = \frac{y}{h} V$$

$$\text{Viscous force } F_v = \mu \frac{du}{dy} \cdot (-\vec{e}_y) \cdot A = \mu \cdot \frac{V}{h} \cdot (-e_x) \cdot A = \frac{\mu A V}{h} \cdot (-e_x)$$

$\frac{e_x}{e_y}$

B. For M: $\Sigma F_x = \vec{F}_V + \vec{F}_T$

$$M \frac{dv}{dt} = F_T - F_v \quad ①$$

For m: $\Sigma F_y = \vec{F}_m + \vec{F}_T$

$$m \frac{dv}{dt} = mg - F_T \quad ②$$

Combine ① and ②

$$M \frac{dv}{dt} + F_v = mg - m\frac{dv}{dt}$$

$$(M+m) \frac{dv}{dt} = mg - F_v$$

$$(M+m) \frac{dv}{dt} = mg - \frac{\mu A V}{h}$$

$$\int_0^V \frac{dv}{mg - \frac{\mu A v}{h}} = \int_0^t \frac{dt}{M+m}$$

$u = mg - \frac{\mu A v}{h}$

$$\int_{mg}^{u} \frac{h}{u} \cdot \frac{du}{-\mu A} = \frac{t}{M+m}$$

$$-\frac{h}{\mu A} \ln(u) \Big|_{u=mg}^{u=u} = \frac{t}{M+m}$$

$$-\frac{h(M+m)}{\mu A} \ln \frac{u}{mg} = t$$

$$-\frac{h(M+m)}{\mu A} \ln \frac{mg - \frac{\mu A V}{h}}{mg} = t$$

C.

$$\exp\left(\frac{-\mu A t}{h(M+m)}\right) = 1 - \frac{\mu A V}{mgh}$$

$$\frac{1 - \exp\left(\frac{-\mu A t}{h(M+m)}\right)}{\mu A} \cdot mgh = V$$

$t \to \infty$

$$V = \frac{mgh}{\mu A}$$

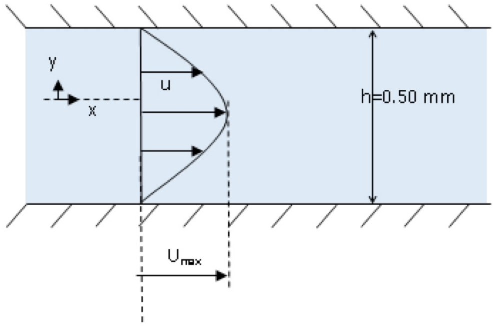
Question 1

The flow profile of water at 15°C between 2 flat plates is given by: $\underline{u} = U_{max}\left[1-\left(\frac{2y}{h}\right)^2\right]\hat{i}$

where U_{max}=0.30 m/s and the water viscosity at 15°C is 1.14×10^{-3}N×s/m².

A. Calculate the shear stress in the liquid and the force per unit area that acts on the top plate.

B. Draw a diagram of the shear stress along the distance between the plates.



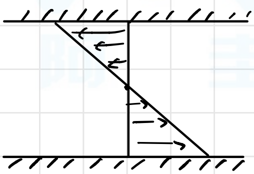
Sol: A. Shear stress $\tau_{yx} = \mu \frac{du}{dy} = \mu \cdot \frac{8U_{max}y}{h^2}$

Force per unit area $F = \tau_{yx} \cdot \vec{A} = \mu \cdot \frac{8U_{max}y}{h^2} \cdot \frac{e_x}{e_y} \cdot (-\vec{e}_y) \cdot A$

$$\frac{F}{A} = \mu \frac{8U_{max}y}{h^2}(-e_x)$$

$$\frac{F(y=\frac{h}{2})}{A} = \mu \frac{4U_{max}}{h} \cdot (-e_x) = 1.14\times10^{-3} N\cdot s/m^2 \cdot \frac{4\times0.3 m\cdot s^{-1}}{0.5\times10^{-3} m} = -2.736 N\cdot m^{-2}$$

B.



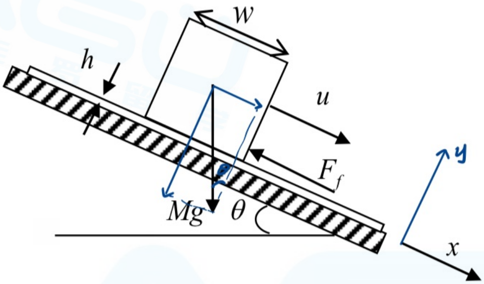
Homework 1

1. A cuboid with an edge size of $0.2m$ and a mass of $2kg$ slides on a slope 30° with regards to the horizon, on an oil film $0.02mm$ thick with viscosity of $\mu = 0.4\,\frac{N\sec}{m^2}$. At time t=0 the cuboid is released.

   A. What is the initial acceleration of the cuboid?

   B. Express the velocity of the cuboid as a function of time.

   C. Plot the velocity as a function of time.



A. Initial accelebration $\vec{a}_i = g\sin\theta \cdot (+e_x)$

B. $\begin{cases} V(y=h) = u \\ V(y=0) = 0 \end{cases} \Rightarrow V(y) = \frac{u}{h}y$

$$\tau_{yx} = \mu\dot{\gamma} = \mu\frac{dv}{dy} = \mu\cdot\frac{u}{h}$$

$$\vec{F} = \tau_{yx}\cdot\vec{A} = \mu\cdot\frac{V}{h}\cdot w^2\cdot\frac{e_x}{e_y}\cdot(-e_y) = \mu\frac{Vw^2}{h}\cdot(-e_x)$$

$$Mg\sin\theta\cdot(+e_x) + \vec{F} = Ma$$

$$\left(Mg\sin\theta - \mu\frac{Vw^2}{h}\right)\cdot(+e_x) = M\frac{dv}{dt}(+e_x)$$

$$Mg\sin\theta - \mu\frac{Vw^2}{h} = M\frac{dv}{dt}$$

$$\frac{dv}{dt} + \mu\frac{Vw^2}{hM} - g\sin\theta = 0$$

$$\frac{dv}{dt} + 0.4\times0.2^2\cdot\frac{V}{2\cdot0.02\times10^{-3}} - 9.8\times\frac{1}{2} = 0$$

$$\frac{dv}{dt} + 400V - 4.9 = 0$$

$$\mu(v) = e^{\int 400dt} = e^{400t}$$

$$(V\cdot e^{400t})' = 4.9e^{400t}$$

$$V\cdot e^{400t} = \frac{4.9e^{400t}}{400} + C$$

$$V = \frac{4.9}{400} + \frac{C}{e^{400t}}$$

Since when $t=0$, $V=0 \Rightarrow C = -\frac{4.9}{400}$

$$V = -\frac{4.9}{400e^{400t}} + \frac{4.9}{400}$$

2. An ice skater weighs $W = 60kg$, skating on one skate for a short time at a velocity of $V = 5m/s$. His weight is held by a thin film of water that melted under the skate.

Calculate the decceleration of the skater at that moment.

Data about the skate:

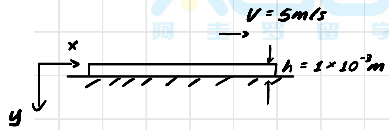
Length and width:

$L = 15cm \;\; w = 5cm$

Width of the fluid film:

$1mm$

Water viscosity $\mu = 0.8\dfrac{N \cdot s}{m^2}$



Sol: No-Slip condition:

$$\begin{cases} u(y=0) = 0 \\ u(y=h) = V \end{cases} \Rightarrow u(y) = \frac{V}{h} y$$

$$\vec{F} = \tau_{yx} \cdot \vec{A} = \mu \dot{\gamma} \cdot \vec{A} = \mu \frac{du}{dy} \cdot \vec{A} = \mu \cdot \frac{V}{h} \cdot (+e_y) L w \cdot \frac{e_x}{e_y} = \frac{\mu V L w}{h} \cdot (+e_x) = 30\,(+e_x) N$$

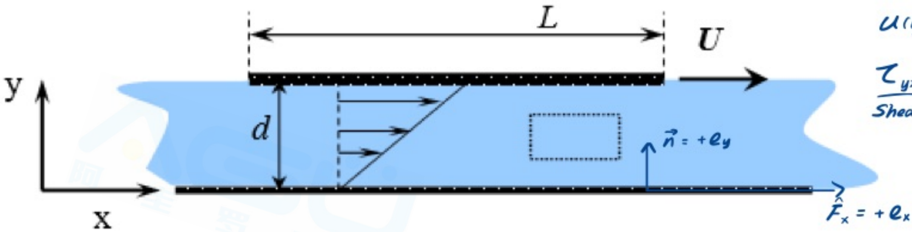
$$\vec{F}_{cubic} = -\vec{F} = 30 \cdot (-e_x) N$$

$$\vec{a} = \frac{\vec{F}_{cubic}}{M} = \frac{-30N}{60kg} = -0.5\,(+e_x) \cdot m/s$$

3.

A flat plate is moving relative to a different flat plate at a steady velocity, $\vec{\mathbf{V}} = U\,\vec{\mathbf{i}}$, see attached figure. Between the plates there is a fluid with viscosity μ. The distance between the plates d is much smaller than their lengths, L. This allows us to assume a linear velocity profile.

$\left.\begin{array}{l} u(y=0)=0 \\ u(y=d)=U \end{array}\right\}$ no-slip condition



$u(y) = \frac{U}{d} y$

$\tau_{yx} = \mu \frac{du}{dy} = \mu \frac{U}{d}$

Shear stress

Which ones of the following sayings are correct? (More than one is true)

a. Maximum rate of strain is at ~~$d/2$~~. d

b. The rate of strain is dictated by U and d, y $\quad u(y) = \frac{U}{d} y$

c. τ_{yx} is not zero $\quad \tau_{yx} = \mu \frac{du}{dy} = \mu \frac{U}{d}$

d. The magnitude of the shear stress is not dependent on y. ✓

e. The direction of forces due to stresses on the rectangular shape in the fluid is the following:

Correct one

f. The smaller is d the ~~smaller~~ ↑ is the shear stress. $\quad$ using $\tau_{yx} = \mu \frac{U}{d}$, if d smaller, τ_{yx} larger.

g. The greater the viscosity of the fluid μ the greater is the shear stress on the plate $\quad \tau_{yx} = \mu \frac{U}{d}$. $\mu\uparrow \rightarrow \tau_{yx}\uparrow$

h. The shear stress the the fluid exerts on the plate does not depend on L. $\quad \tau_{yx} = \mu \frac{U}{d}$

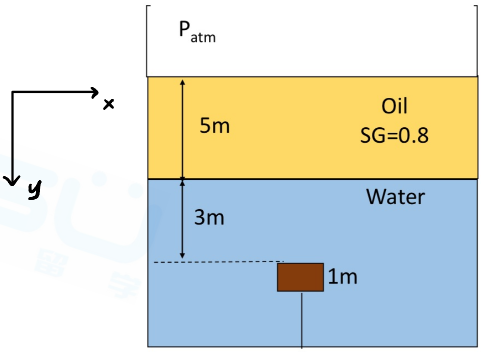
i. The required force to move the upper plate does ~~not~~ depend on L.

$F = \tau A = \tau L^2$

b. c. d. g. h

Tutorial 2

1. A cube with an edge of 1m, made of wood, is held in a cuboid by a rope as depicted in the figure. Calculate the force that the water inflicts on the bottom of the cube and the tension in the rope. The specific gravity of wood is SG=0.77.



Sol: $-\nabla P + \rho g = 0$

$$\begin{cases} -\frac{\partial P}{\partial x} + \rho g_x = 0 \\ -\frac{\partial P}{\partial y} + \rho g_y = 0 \\ -\frac{\partial P}{\partial z} + \rho g_z = 0 \end{cases} \Rightarrow g_y = g$$

$$\frac{dP}{dy} = \rho g \qquad\qquad \frac{dP}{dy} = \rho g$$

$$\int_{P_0}^{P} dP = \rho g \int_0^y dy \qquad\qquad \int_{P_0} dP = \rho_{oil} g \int_0^h dy + \rho_{water} g \int_h^y dy$$

$$P - P_0 = \rho g y \qquad\qquad P_g = 0.8\rho g h + \rho g (y-h) \quad (\text{if } y \geq 5m,\ h = 5m)$$

$$P_g = \rho g y \quad (\text{if } y < 5m)$$

Force that the water inflict on the bottom

$F_b = \int -P_g dA = [-0.8\rho g h - \rho g (y-h)] \cdot (+e_y) \cdot A^2$

$y = 9m,\ A^2 = 1m^2,\ \rho = 998\ kg/m^3,\ h = 5m,\ g = 9.8\ m \cdot s^{-2}$

$F_b = -78243.2N$

Force that the water inflict on the top

$F_t = \int -P_g dA = [-0.8\rho g h - \rho g (y-h)] \cdot (-e_y) \cdot A^2$

$y = 8m,\ A^2 = 1m^2,\ \rho = 998\ kg/m^3,\ h = 5m,\ g = 9.8\ m \cdot s^{-2}$

$F_t = 68462.8N$

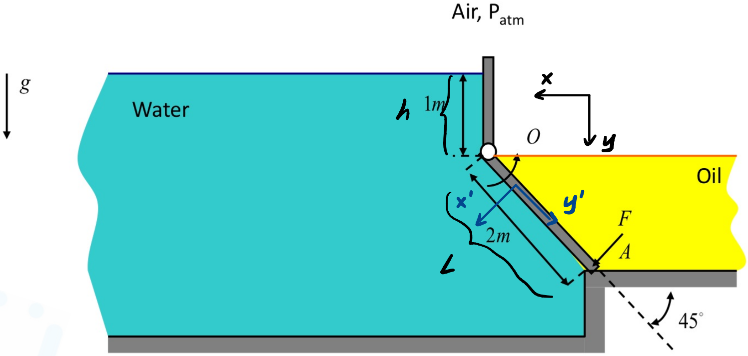
记得计算重力

$F_w = mg = \rho_w V g = 0.77 \times 998 kg/m^3 \times 1m^3 \times 9.8\ m \cdot s^{-2} = 7530.908N$

Tension Force, $T = F_b + F_t + F_w = 2249.492N$

2. A gate between two big reservoirs of water and oil can be opened by rotating around the axis at point O counterclockwise. The width of the gate along an axis at O is 2m. Calculate the force F that is needed to apply at point A in order to keep the gate closed. The density of water is $\rho = 998 kg/m^3$ and of oil $\rho = 1020 kg/m^3$

Is the system stable?
In other words, How will a small opening of the gate affect the situation? Will it tend to close or open even more? Please explain.



Sol: For water:

$$-\nabla P + \rho g = 0$$

$$\begin{cases} -\frac{\partial P}{\partial x} + \rho g_x = 0 \\ -\frac{\partial P}{\partial y} + \rho g_y = 0 \longrightarrow g_y = g \\ -\frac{\partial P}{\partial z} + \rho g_z = 0 \end{cases}$$

$$\frac{dP}{dy} = \rho g$$

$$\int_{P_0}^{P} dP = \rho g \int_{-h}^{y} dy$$

$$P - P_0 = \rho g (y + h)$$

$$P_g = \rho g (y + h)$$

Since $y = y' \cos 45°$

$$P_g = \rho g y' \cos 45° + \rho g h$$

$$\vec{M}_{water} = \int_0^L \vec{r} \times d\vec{F}$$

$$= \int_0^L \vec{r} \times (-P_g d\vec{A})$$

$$= -\int_0^L \vec{r} \times (\rho g y' \cos 45° + \rho g h) \cdot (+e_x) w dy'$$

$$= -\int_0^L \begin{vmatrix} e_x & e_y & e_z \\ 0 & y' & 0 \\ (\rho g y' \cos 45° + \rho g h) w dy' & 0 & 0 \end{vmatrix}$$

$$= \int_0^L (\rho g y' \cos 45° + \rho g h) y' \cdot w dy' \ (+e_z)$$

$$= \left[\frac{\rho g y'^3}{3} \cos 45° + \frac{\rho g y'^2}{2} h \right]_{y'=0}^{y'=L} \cdot w \cdot (+e_z)$$

$$= 38002.8991 w$$

For oil:

$$\int_{P_0}^{P} dP = \rho_{oil} g \int_0^y dy$$

$$P - P_0 = \rho_{oil} g y$$

$$P_g = \rho_{oil} g y' \cos 45°$$

$$\vec{M}_{oil} = \int_0^L \vec{r} \times d\vec{F}$$

$$= \int_0^L \vec{r} \times (-P_g d\vec{A})$$

$$= \int_0^L \vec{r} \times (-\rho_{oil} g y' \cos 45°) \cdot (-e_x) w dy'$$

$$= \int_0^L \vec{r} \times (\rho_{oil} g y' \cos 45°) \cdot (+e_x) w dy'$$

$$= \int_0^L \begin{vmatrix} e_x & e_y & e_z \\ 0 & y' & 0 \\ \rho_{oil} g y' \cos 45° w dy' & 0 & 0 \end{vmatrix}$$

$$= \int_0^L -\rho_{oil} g y'^2 \cos 45° w dy' = \left[-\frac{\rho_{oil} g (y')^3}{3} \cos 45° w \right]_{y'=0}^{y'=L}$$

$$= -18848.63836 w$$

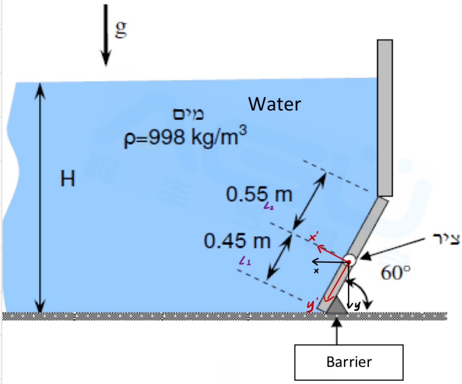
Since $\vec{M}_{water} + \vec{M}_{oil} + \vec{M}_F = \Sigma M = 0$

$$\vec{M}_F = 19154.26074 w \ (-e_x)$$

Homework 2

注意坐标轴原点必须是转动轴心

1. A gate with a length of 1m and width of w regulates the water inside a reservoir. The gate can only rotate clockwise around a rotation axis located 0.45 m from the bottom of the reservoir, as depicted in the figure. Calculate the height H, above which the gate will open spontaneously.



Sol: $-\nabla p + \rho g = 0$

$$\begin{cases} -\dfrac{\partial p}{\partial x} + \rho g_x = 0 \\ -\dfrac{\partial p}{\partial y} + \rho g_y = 0 \quad \longrightarrow \quad g_y = g \\ -\dfrac{\partial p}{\partial z} + \rho g_z = 0 \end{cases}$$

$$\frac{dp}{dy} = \rho g$$

$$\int_{P_0}^{P} dP = \rho g \int_{-(H-L_1\sin 60^\circ)}^{y} dy$$

$$P - P_0 = \rho g (y + H - L_1 \sin 60^\circ)$$

$$P_g = \rho g (y + H - L_1 \sin 60^\circ)$$

since $y = y'\cos 30^\circ$

$$P_g = \rho g (y'\cos 30^\circ + H - L_1 \sin 60^\circ)$$

$$\vec{F} = \int -P_g\, dA = \int -P_g w\, dy' \cdot (+e_x)$$

$$\vec{M} = \int \vec{r} \times d\vec{F}$$

$$\vec{M}_1 = \int \vec{r}_1 \times d\vec{F}_1 = \int \begin{vmatrix} e_x & e_y & e_z \\ 0 & y' & 0 \\ -P_g w dy' & 0 & 0 \end{vmatrix}$$

(Lower one)

$$= \int_0^{L_1} P_g w y' dy' = \int_0^{L_1} \rho g w [y'^2 \cos 30^\circ + (H - L_1 \sin 60^\circ) y'] dy'$$

$$= \rho g w \left[\frac{(y')^3}{3} \cos 30^\circ + \frac{(y')^2}{2} \cdot (H - L_1 \sin 60^\circ) \right] \Bigg|_{y'=0}^{y'=L_1}$$

$$= \rho g w \left[\frac{L_1^3}{3} \cos 30^\circ + \frac{L_1^2}{2} \cdot (H - L_1 \sin 60^\circ) \right]$$

$$\vec{M}_2 = \int \vec{r}_2 \times d\vec{F}_2 = \int \begin{vmatrix} e_x & e_y & e_z \\ 0 & -y' & 0 \\ -P_g w dy' & 0 & 0 \end{vmatrix} = \int_0^{-L_2} -P_g w y' dy'$$

$$= \int_0^{-L_2} -\rho g w [y'^2 \cos 30^\circ + (H - L_1 \sin 60^\circ) y'] dy'$$

$$= -\rho g w \left(\frac{(y')^3}{3} \cos 30^\circ + \frac{(y')^2}{2} (H - L_1 \sin 60^\circ) \right) \Bigg|_{y'=0}^{y'=-L_2}$$

$$= \rho g w \left[\frac{L_2^3}{3} \cos 30^\circ - \frac{L_2^2}{2} (H - L_1 \sin 60^\circ) \right]$$

To be stable $\Sigma M = \vec{M}_1 + \vec{M}_2 = 0$

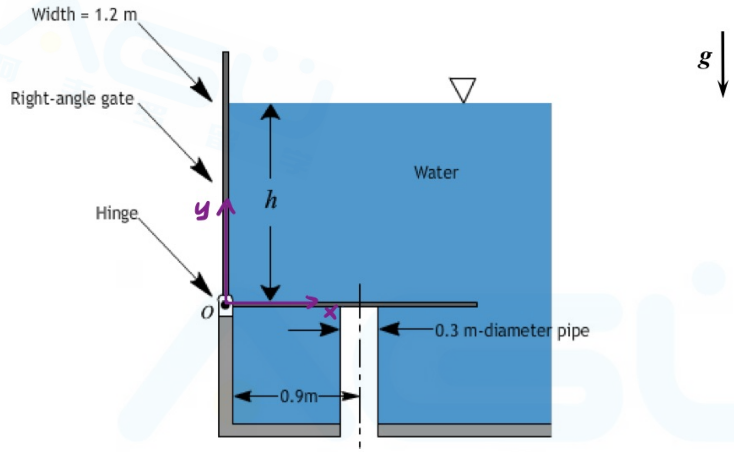
$$\rho g w \left[\frac{L_1^3}{3} \cos 30^\circ + \frac{L_1^2}{2} (H - L_1 \sin 60^\circ) + \frac{L_2^3}{3} \cos 30^\circ - \frac{L_2^2}{2} (H - L_1 \sin 60^\circ) \right] = 0$$

$$\left(\frac{L_1^2}{2} - \frac{L_2^2}{2}\right) H = -\frac{L_1^3}{3} \cos 30^\circ + \frac{L_1^3}{2} \sin 60^\circ - \frac{L_2^3}{3} \cos 30^\circ - \frac{L_1 L_2^2}{2} \sin 60^\circ$$

$$H = 1.8764 m$$

2. A gate with a shape of Γ(right angle gate) with a width of 1.2(into the paper) and a negligible mass can turn around an axis point O without friction as depicted in the figure. The horizontal part of the gate covers a drainage pipe full of air, which is 0.9 m from the vertical part of the gate. The mass and volume of the gate can be neglected. The density of water is $\rho_W = 998 \frac{kg}{m^3}$.

a. Find the resultant force that the water exerts on the vertical side of the gate.(using gauge pressure)
b. Find the resultant force which the water and the air exert on the horizontal side of the gate.
c. Find the minimum height h for which the gate will open and the water will flow into the drainage pipe.



Sol: a. $-\nabla P + \rho g = 0$

$$\begin{cases} -\frac{\partial P}{\partial x} + \rho g_x = 0 \\ -\frac{\partial P}{\partial y} + \rho g_y = 0 \\ -\frac{\partial P}{\partial z} + \rho g_z = 0 \end{cases}$$

$g_y = -g$

$\frac{dP}{dy} = -\rho g$

$\int_{P_0}^{P} dP = -\rho g \int_h^y dy$

$P_g = \rho g (h-y)$

$$\vec{F}_v = \int -P_g \, dA = \int_0^h \rho g (y-h) w \, dy = \rho g \left(\frac{y^2}{2} - hy\right) w \Big|_{y=0}^{y=h} = \rho g \left(\frac{h^2}{2} - h^2\right) w = -\rho g \cdot \frac{h^2}{2} w \, (+e_x)$$

b. $\vec{F}_h = \int -P_g \, dA = \rho g h w \cdot \pi r^2 \cdot (-e_y)$

c. Vertical: $\vec{M}_1 = \int \vec{r} \times d\vec{F}$

$$= \int \begin{vmatrix} e_x & e_y & e_z \\ 0 & y & 0 \\ \rho g (y-h) w \, dy & 0 & 0 \end{vmatrix}$$

$$= \int_0^h \rho g w (y^2 - hy) \, dy = \rho g w \left(\frac{y^3}{3} - \frac{hy^2}{2}\right) \Big|_{y=0}^{y=h}$$

$$= \rho g w \cdot \frac{h^3}{6} \cdot (-e_z)$$

$r = \frac{d}{2}$

Horizontal: $\vec{M}_2 = \vec{r} \times \vec{F} = 0.9 \cdot (+e_x) \times \rho g h w \cdot \pi r^2 \cdot (-e_y) = -\frac{0.9}{4} \rho g h \pi d^2 e_z$

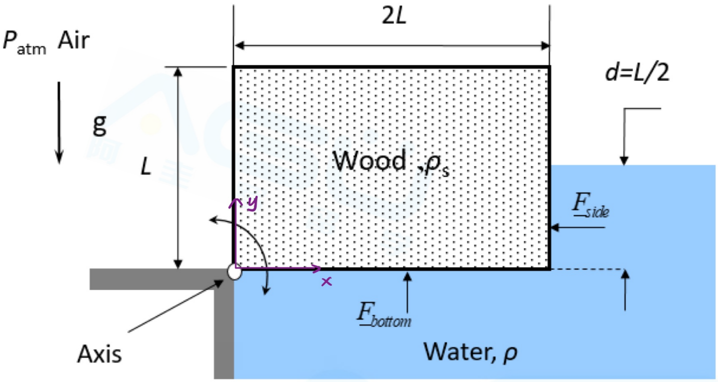
$\Sigma M = \vec{M}_1 + \vec{M}_2 = 0$

$\rho g w \left(-\frac{h^3}{6} - 0.9 h \cdot \pi \cdot \frac{d^2}{4}\right) = 0$

$h = 0.564 m$

The minimum height is 0.564m.

3. A log(with perpendicular width of W) has a rectangular cross section with size $L\times 2L$, can rotate around an axis of rotation which passes through one of its edges. The log is at equilibrium
   A. Find the resultant forces that the water inflicts on the log's wet faces.
   B. Find the center of pressure on each of the wet faces.
   C. In case of negligible friction in the axis, find the ratio of the density of the log and the water, ρ_s/ρ.



Sol: A. $-\nabla P + \rho g = 0$

$$\begin{cases} -\frac{\partial P}{\partial x} + \rho g_x = 0 \\ -\frac{\partial P}{\partial y} + \rho g_y = 0, \quad g_y = -g \\ -\frac{\partial P}{\partial z} + \rho g_z = 0 \end{cases}$$

$$-\frac{\partial P}{\partial y} = \rho g$$

$$\int_{P_0}^{P} -dP = \rho g \int_{\frac{L}{2}}^{y} dy$$

$$P_0 - P = \rho g\left(y - \frac{L}{2}\right)$$

$$P_g = \rho g\left(\frac{L}{2} - y\right)$$

$$\vec{F}_{side} = \int -P_g\,dA$$
$$= \int_0^{\frac{L}{2}} \rho g\left(y - \frac{L}{2}\right) w\,dy\,(+e_x)$$
$$= \left[\rho g\left(\frac{y^2}{2} - \frac{Ly}{2}\right) w\right]_{y=0}^{y=\frac{L}{2}} (+e_x)$$
$$= \rho g \frac{L^2}{8} w \cdot (-e_x)$$

$$\vec{F}_{bottom} = \int -P_g\,dA$$
$$= \int_0^{2L} -\rho g \frac{L}{2} w\,dx \cdot (-e_y)$$
$$= \left[\rho g \cdot \frac{L}{2} w x\right]_{x=0}^{x=2L} \cdot (-e_y)$$
$$= \rho g L^2 w \cdot (-e_y)$$

B. Centre of pressure: (Vertical side)

$$\int \vec{r} \times d\vec{F}_v = \vec{r}\,' \times \vec{F}_v$$

$$\int \begin{vmatrix} e_x & e_y & e_z \\ 0 & y & 0 \\ \rho g\left(y - \frac{L}{2}\right) w\,dy & 0 & 0 \end{vmatrix} = y' \cdot (+e_y) \times \rho g \frac{L^2}{8} w \cdot (-e_x)$$

$$\int_0^{\frac{L}{2}} \rho g\left(\frac{L}{2} y - y^2\right) w\,dy \cdot (+e_z) = y' \cdot \rho g \frac{L^2}{8} w \cdot (+e_z)$$

$$\cancel{\rho g}\left(\frac{L}{4} y^2 - \frac{y^3}{3}\right)\cancel{w}\Big|_{y=0}^{y=\frac{L}{2}} \cancel{\cdot (+e_z)} = y' \cdot \cancel{\rho g} \frac{L^2}{8} \cancel{w} \cdot \cancel{(+e_z)}$$

$$\frac{L^3}{16} - \frac{L^3}{24} = y' \cdot \frac{L^2}{8}$$

$$y' = \frac{L}{6}$$

Center of pressure: $\left(2L, \frac{L}{6}\right)$

In the horizontal side, the pressure is constant at every point.

C. Buoyancy force 浮力

$$\vec{F}_B = -\oint_A p\vec{n}\,dA$$
$$= \iint_{bottom} -P_g \vec{n}\,dA$$
$$= \int_0^{2L} -\rho g \frac{L}{2} \cdot (-e_y) w\,dx$$
$$= \rho g \frac{L}{2} x w \cdot (+e_y) \Big|_{x=0}^{x=2L}$$
$$= \rho g L^2 w \cdot (+e_y)$$

$$\vec{F}_g = \rho_s g \cdot 2L^2 w \cdot (-e_y)$$

$$\vec{F}_g + \vec{F}_B = 0$$

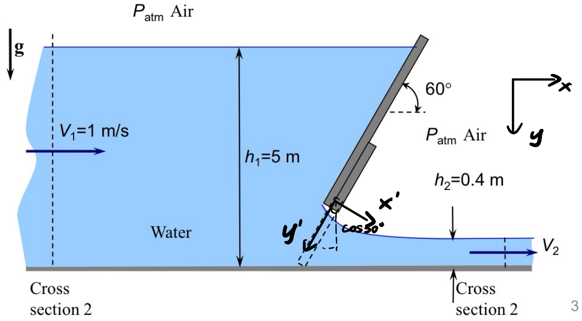
$$\rho_s \cancel{g} \cdot 2\cancel{L^2}\cancel{w} \cdot (-e_y) + \rho \cancel{g}\cancel{L^2}\cancel{w}(+e_y) = 0$$

$$\frac{\rho_s}{\rho} = 0.5$$

Tutorial 3

1. A. Water in an open channel is blocked by a sluice gate. A closed gate is depicted by a dashed line. Calculate the horizontal force per unit width of the gate that the water exert on the gate in 2 cases: (a) the gate is closed (no flow); (b) the gate is open(steady state flow). Assume that the flow at cross section 1 and 2 is uniform and incompressible. Also assume that the pressure distribution at cross section 1 and 2 are hydrostatic and that the shear forces from the water on the gate are negligible. g=9.81 m/s^2 , water density ρ=999 kg/m^3 and the angle between the gate and the horizontal plane is ϑ=60°.

B. A young engineer claims, without calculating the force in A that the force that the water exerts on the gate is greater when the gate is open. Do you think he is right? Please explain.



3

Sol: A. (1) The gate is closed

$$\vec{F} = \int -P\,d\vec{A}$$

$$-\nabla P + \rho g = 0$$

$$\Rightarrow -\frac{\partial P}{\partial y} + \rho g_y = 0, \quad g_y = g$$

$$\frac{dP}{dy} = \rho g$$

$$\int_{P_{atm}}^{P} dP = \rho g \int_0^y dy$$

$$P - P_{atm} = \rho g y$$

$$P_g = \rho g y$$

Since $y = y'\cos 30° = \frac{\sqrt{3}}{2} y'$

$$P_g = \frac{\sqrt{3}}{2}\rho g y'$$

$$\vec{F} = \int_0^L -\frac{\sqrt{3}}{2}\rho g y' \cdot (-e_{x'})\, w\,dy'$$

$$\vec{F} = (+e_{x'})\cdot \int_0^L \frac{\sqrt{3}}{2}\rho g y' \cdot w\,dy'$$

$$\vec{F} = (+e_{x'})\left[\frac{\sqrt{3}}{4}\rho g (y')^2 w\right]_{y'=0}^{y'=L}$$

$$\frac{\vec{F}}{w} = (+e_{x'})\cdot \frac{\sqrt{3}}{4}\rho g L^2$$

$$L\cos 30° = h_1$$

$$L = \frac{2h_1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}h_1$$

$$\frac{\vec{F}}{w} = (+e_{x'})\cdot \frac{\cancel{\sqrt{3}}}{\cancel{4}}\rho g \cdot \frac{\cancel{4}}{3}\cdot\frac{h_1^2}{\sqrt{3}} = \frac{1}{\sqrt{3}}\rho g h_1^2 (+e_{x'})$$

Horizontal Force acts on the gate:

$$\frac{\vec{F_H}}{w} = \frac{\vec{F}}{w}\cdot\cos 30° = \frac{1}{2}\rho g h_1^2 (+e_x) = 122502.375 \text{ N/m}$$

(2) The gate is open

Momentum Balance: $\Sigma\vec{F} = \oiint_{c.s} \rho\vec{v}\cdot(\vec{v}\cdot\vec{n})\,dA + \cancel{\frac{\partial}{\partial t}\iiint_{c.v}\rho\vec{v}\cdot dv}\to 0$

$$\Sigma F_x = \oiint_{c.s}\rho v_x(\vec{v}\cdot\vec{n})\,dA$$

$$\Sigma F_x = \iint_{c.s1}\rho V_1(-V_1)\cdot dA + \iint_{c.s2}\rho V_2\cdot(V_2)\,dA$$

$$= (-\rho V_1^2 A_1 + \rho V_2^2 A_2)\cdot(+e_x)$$

$$= (-\rho V_1^2 w h_1 + \rho V_2^2 w h_2)\cdot(+e_x)$$

$$\Sigma\vec{F_x} = \vec{F_{R1}} + \vec{F_{R2}} + \vec{R_x}$$

$$\vec{F_{R1}} = \int -P_g\,dA_1 = (-e_x)\int_0^{h_1} -\rho g y w\,dy = \frac{\rho g w h_1^2}{2}\cdot(+e_x)$$

$$\vec{F_{R2}} = \int -P_g\,dA_2 = (+e_x)\int_0^{h_2} -\rho g y w\,dy = \frac{\rho g w h_2^2}{2}\cdot(-e_x)$$

Mass Balance:

$$\cancel{\frac{\partial}{\partial t}\iiint_{c.v}\rho\,dt}\to 0 + \oiint_{c.s}\rho\vec{v}\cdot d\vec{A} = 0$$

$$\int_0^{h_1}\rho V_1\cdot(+e_x)\cdot(-e_x)\,w\,dy + \int_0^{h_2}\rho V_2\cdot(+e_x)\cdot(+e_x)\,w\,dy = 0$$

$$-V_1 h_1 + V_2 h_2 = 0 \Rightarrow V_2 = \frac{V_1 h_1}{h_2} = \frac{1\,m\cdot s^{-1}\times 5m}{0.4m} = 12.5\,m\cdot s^{-1}$$

$$\vec{R_x} = \Sigma F_x - \vec{F_{R1}} - \vec{F_{R2}} = -64275.8598\,w \text{ N}$$

$$\frac{\vec{R_x}}{w} = 64275.8598\cdot(-e_x)\cdot N\cdot m^{-1}$$