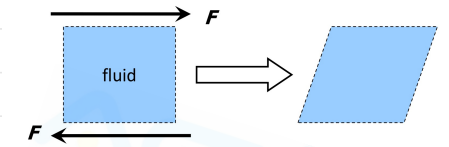
# Lecture 1

**Fluid:** A substance in the Liquid or Gas phase is refer to be fluid.

**Mechanics:** is the branch of physical science that deals with the behaviour of *stationary* and *moving* bodies when subjected to force.

## Fluids:

**Def 1.** Fluid is a substance that continuously deforms (i.e. flows) under an applied shear stress.




**Def 2.** Substances that are unable to support shear stresses in static equilibrium and possess the ability to flow.

**Def 3.** According to Rheology there is no clear distinction between fluid and solid. Fluidity → Debora number: $De = \frac{t_{rel}}{t_{exp}}$

$t_{rel}$ → Relaxation time

$t_{exp}$ → Time of experiment

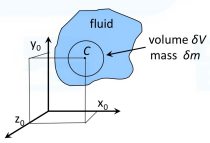
$De \to \infty$ Solid-like

$De \approx 1$ Intermediate

$De \approx 0$ Fluid-like

## The Concept of "Continuum".

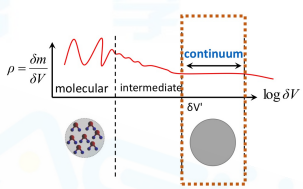
The material is treated as continuum, meaning that it can be continually sub-divided into infinitesimal elements with properties (such as density, pressure, temp. etc) being those of the bulk material.




example: volume of $10^{-12}$ m$^3$ contains $2.5 \cdot 10^{13}$ gas molecules!!

**Density at point C is defined as:**

$$\rho(x,y,z) = \lim_{\delta V \to \delta V'} \frac{\delta m}{\delta V}$$




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## Fluid Velocity

$$\vec{V} = \vec{V}(x,y,z,t) = V_x \hat{e}_x + V_y \hat{e}_y + V_z \hat{e}_z$$

($\hat{i}$, $\hat{j}$, $\hat{k}$)

$$V = |\vec{V}| = (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}}$$

Steady-State (Velocity is constant): $\vec{V} = \vec{V}(x,y,z)$

$$\frac{\partial \vec{V}}{\partial t} = 0$$

Plane Flow: $\vec{V} = \vec{V}(x,y,t) = V_x(x,y,t)\,\hat{i} + V_y(x,y,t)\,\hat{j}$

Unidirectional Flow: $\vec{V} = V_x(y,z)\,\hat{i}$

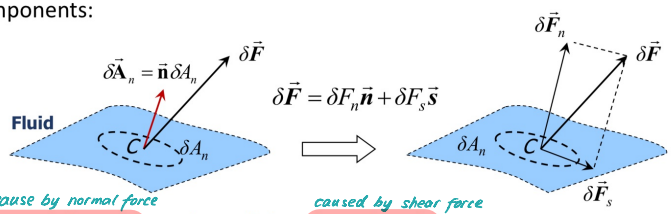
## Pressure and Stress

Body force (inside the item): acting throughout the volume of fluid element. Care about Mass (example: gravity)

Surface force: internal forces acting across surfaces of fluid elements. Can be decomposed:

Pressure (normal), Shear (tangential) Care about Volume.

We can decompose the **surface force** acting on a surface of fluid element at point *C* into **normal** and **tangential** components:




We can then define normal stress ($\sigma$) and tangential or shear stress ($\tau$) as:

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n}, \quad \tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_s}{\delta A_n}$$

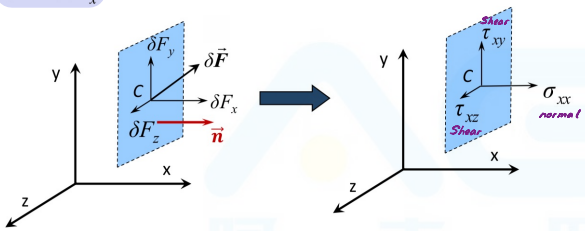
Stresses exerted on $\delta\vec{A}_n = \vec{n}\delta A_n$

**Due to this definition, $\sigma$ and $\tau$ depend on choice of the surface orientation!**

**We look for an alternative definition of stress exerted by the fluid at point *C* that does not depend on the choice of surface passing though *C* :**

**For Example we choose plane with normal at x direction:**

$\delta\vec{A} = \vec{i}\delta A_x$




$$\delta\vec{F} = \delta F_x\vec{i} + \delta F_y\vec{j} + \delta F_z\vec{k}$$

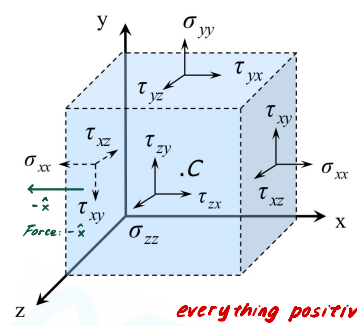
$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x}$$

$$\tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x}$$

$$\tau_{xz} = \lim_{\delta A_x \to 0} \frac{\delta F_z}{\delta A_x}$$

**double index:**
1st: surface orientation
2nd: force direction

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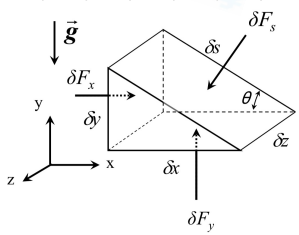



everything positive.

All stress components here are positive!

Reversing the direction of either normal or force: revert sign

Flipping both: preserves it!




Static Fluid: Shear Stress absent. 缺席

Normal Stress: $\Sigma\vec{F} = 0$

$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$ (Pascal Law)

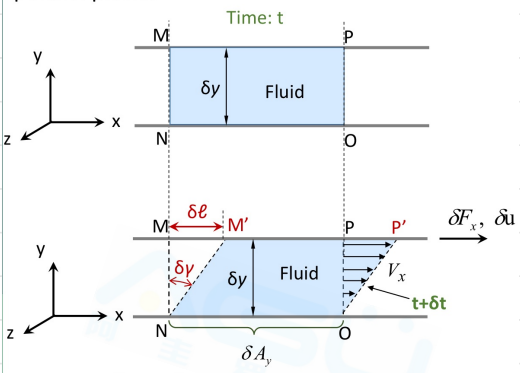
(Pressure uniform everywhere.)

$$P = -\frac{1}{3}\sum_{i=1}^{3}\sigma_{ii} = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

三个为零

# Viscosity → Shear Stress, Shear / Strain / Deformation Rate

For a fluid encapsulated between two long parallel plates:




$$\frac{\delta\gamma}{\delta t} = \frac{\delta u}{\delta y} = \dot{\gamma} = \text{Const}$$

Shear stress: direction: $\hat{y}$, force: $\hat{x}$

$$\tau = \tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y}$$

Strain(shear) rate: ↗ Angle

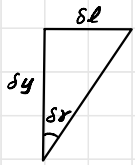
$$\lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \frac{d\gamma}{dt} = \dot{\gamma}$$

We can't describe the change with time, thus we use measurable parameters:

$$\delta\ell \approx \delta u \delta t; \quad \delta\ell = \delta y \tan\delta\gamma \approx \delta y \delta\gamma$$

$$\frac{\delta\gamma}{\delta t} = \frac{\delta u}{\delta y} = \dot{\gamma} = \text{Const}$$

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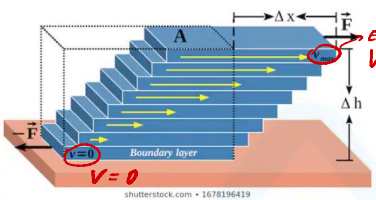

$\delta\ell = \delta u \cdot \delta t$

$\frac{\delta\ell}{\delta y} = \tan\delta\gamma$ if it's very small then $\tan\delta\gamma = \delta\gamma$

$\delta\ell = \delta y \cdot \delta\gamma$

$\delta y \cdot \delta\gamma = \delta u \cdot \delta t$

$\frac{\delta\gamma}{\delta t} = \frac{\delta u}{\delta y} = \dot{\gamma}$




Exactly the velocity of the surface

$V_{max}$

Shear Stress exists on each layer.

V = 0

In any unidirectional flow

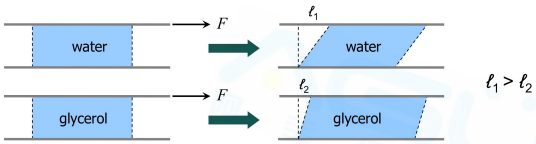
$$\frac{d\gamma}{dt} = \frac{dV_x}{dy}$$

- Fluids for which shear stress $\tau = \tau_{yx}$ is linearly proportional to strain (shear) rate $\dot{\gamma}$ are called **Newtonian fluids**, e.g., water, gases, oils, alcohols, naphtha, etc. and the proportion constant is the **viscosity.**

## Viscosity - Newtonian Fluid

Shear stress ⇐⇒ Strain rate

$\tau_{yx}$ ⇐⇒ $\dot{\gamma}$




We then say that glycerol is more viscous than water!

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## Newtonian Fluid (proportionality coefficient μ)

$\tau = \mu\dot{\gamma}$

↳ viscosity (How much material will resist the force.)

if $\mu \to \infty$, solid-like

Dimensions and units of viscosity:

$$\tau = \left[\frac{F}{L^2}\right], \dot{\gamma} = \left[\frac{1}{T}\right] \Rightarrow \mu = \left[\frac{F \cdot T}{L^2}\right] = \left[\frac{M}{L \cdot T}\right]$$

In the SI system we use the units of Pa·s,

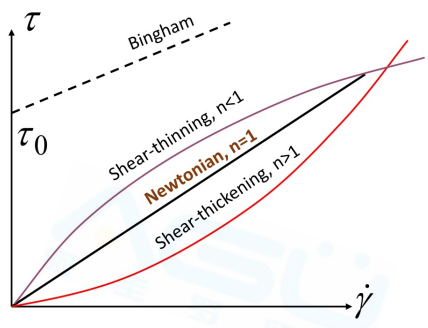
The viscosity of water at room temp.: 0.01P = 1cP = $10^{-3}$ Pa·s (1 P=0.1 Pa·s)

**Kinematic viscosity:** $\nu = \frac{\mu}{\rho} = \left[\frac{M}{L \cdot T}\frac{L^3}{M}\right] = \left[\frac{L^2}{T}\right]$

Kinematic Viscosity $\nu = \frac{\mu}{\rho} = \left[\frac{M}{L \cdot T} \cdot \frac{L^3}{M}\right] = \left[\frac{L^2}{T}\right]$

## Viscosity – non-Newtonian Fluid 非牛顿流体

- Fluids for which the above does not hold are called **non-Newtonian fluids**, e.g., polymer solutions, emulsions, dispersions, gels, etc.




- Generalized Newtonian fluid:

$$\tau = m\dot{\gamma}^n = \underbrace{m|\dot{\gamma}|^{n-1}}_{\mu_{app}(\dot{\gamma})}\dot{\gamma}$$

$n<1: \dot{\gamma}\uparrow \ \mu_{app}\downarrow$ - shear - thinning 剪切稀化

$n>1: \dot{\gamma}\uparrow \ \mu_{app}\uparrow$ - shear - thickening

- Bingham fluid: $\tau = \tau_0 + \mu\dot{\gamma}$

- Viscoelastic fluid: $\tau = G\gamma + \mu\dot{\gamma}$

(Kelvin model)



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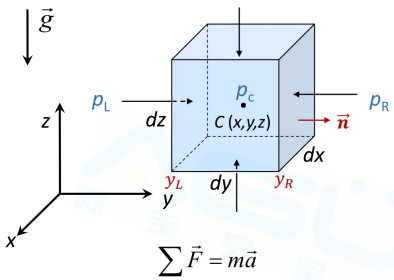
# Lecture 2

**Static fluid**: shear stress absent; the only surface force is due to pressure. (normal force)

(or fluid in a rigid-body motion)

In a static fluid: $\Sigma \vec{F} = 0$

In a fluid in a rigid-body motion: $\Sigma \vec{F} = m\vec{a}$




Body Force: $d\vec{F}_g = \vec{g}\,dm = \vec{g}\rho dV = \vec{g}\rho\,dx\,dy\,dz$ (density)

Force Balance (y-direction)

$p_R = p_c$

$d\vec{F}_p = \ominus p d\vec{A}$

$d\vec{F}_{p,R} = -p_R d\vec{A}$

$d\vec{F}_{p,L} = -p_L d\vec{A}$

$\Sigma \vec{F} = d\vec{F}_{p,R} + d\vec{F}_{p,L}$

$d\vec{F}_s = -\left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)_c (dx\,dz)(-\vec{j}) + \left(p + \frac{\partial p}{\partial y}\cdot\frac{dy}{2}\right)_c (dx\,dz)\vec{j}$ $(-1)$

($p_c$ and $d\vec{F}_{p,L}$; $d\vec{F}_{p,R}$; $p_c$)

$= -\left(\frac{\partial p}{\partial y}\right)_c dx\,dy\,dz\,\vec{j}$

Assuming we know what is $P_c$, if we expand the pressure at $y_R$ and $y_L$ using Tylor series:

$$p_R = p_c + \left(\frac{\partial p}{\partial y}\right)_c (y_R - y) + \cdots = p_c + \left(\frac{\partial p}{\partial y}\right)_c \frac{dy}{2} + \cdots$$

$$p_L = p_c + \left(\frac{\partial p}{\partial y}\right)_c (y_L - y) + \cdots = p_c - \left(\frac{\partial p}{\partial y}\right)_c \frac{dy}{2} + \cdots$$

(minus)

## Fluid static equation

$$-\nabla p + \rho(\vec{g} - \vec{a}) = 0$$

## Victorian equation

$$-\nabla P + \rho g = 0$$

$$\vec{\nabla} = \frac{\partial}{\partial x}e_x + \frac{\partial}{\partial y}e_y + \frac{\partial}{\partial z}e_z$$

$$\vec{\nabla} p = \frac{\partial p}{\partial x}e_x + \frac{\partial p}{\partial y}e_y + \frac{\partial p}{\partial z}e_z$$

$$-\frac{\partial p}{\partial x} + \rho g_x = 0$$

$$-\frac{\partial p}{\partial y} + \rho g_y = 0$$

$$-\frac{\partial p}{\partial z} + \rho g_z = 0$$

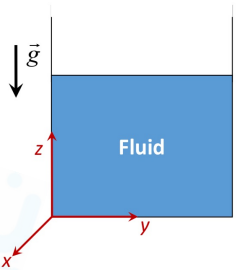
$d\vec{F}_s = -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right) dx\,dy\,dz = -(\nabla p)_c\,dV$

$\Sigma \vec{F} = m\vec{a}$

$d\vec{F} = d\vec{F}_s + d\vec{F}_g = (-\nabla p + \rho\vec{g})\,dV = \vec{a}\rho\,dV$

$\frac{d\vec{F}}{dV} = \vec{a}\rho = -\nabla p + \rho\vec{g}$

**Basic application of the Fluid statics equation:**




$-\frac{\partial p}{\partial z} + \rho g_z = 0$

$-\frac{\partial p}{\partial z} - \rho g = 0$

$\frac{dp}{dz} = -\rho g$

$\vec{g}_z = -\vec{g}$

$$\underbrace{-\frac{\partial p}{\partial x} = 0,\quad -\frac{\partial p}{\partial y} = 0}_{p = p(z)},\qquad -\frac{\partial p}{\partial z} - \rho g = 0 \quad (g = |\vec{g}|)$$

$$\boxed{\frac{dp}{dz} = -\rho g}$$

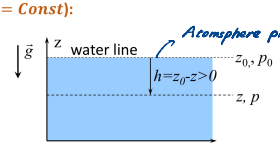
In a static fluid (or in a steady rigid-body motion) the pressure changes (decreases) only with elevation ($z$)!

$\int_{p_0}^{p} dp = -\rho g \int_{z_0}^{z} dz$

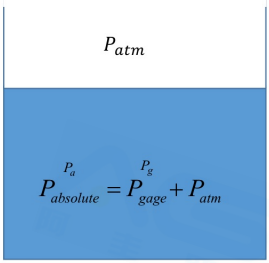
$p - p_0 = -\rho g\,(z - z_0)$ $(-h)$

$\boxed{p = p_0 + \rho g h}$

**Pressure change in an incompressible fluid ($\rho = Const$):**



- **Absolute pressure is measured relative to a perfect vacuum, whereas gage pressure is measured relative to the atmospheric pressure.**
- $P_{gage}$ is the Pressure measured relative to a reference pressure. The reference pressure is arbitrary. Usually it is taken to be the atmospheric pressure.

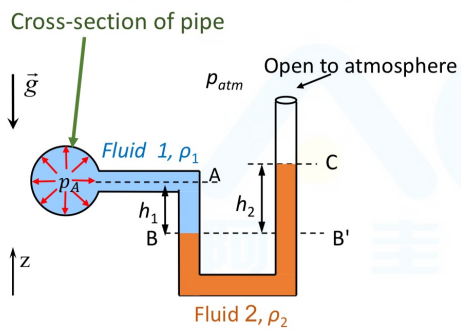



$$P_{absolute}^{P_a} = P_{gage}^{P_g} + P_{atm} \quad \Leftrightarrow \quad p_{gage} = p_{abs} - p_{atm},$$

$$p_{atm} = 101.3 \text{ kPa}, 760 \text{ mmHg}$$

**Pressure measurement by manometers is based on:**

$$p = p_{atm} + \rho g h$$




$$p_B = p_A + \rho_1 g h_1$$

$$p_{B'} = \rho_2 g h_2 + p_c$$

$$p_c = p_{\text{atm}}$$

$$p_B = p_{B'}$$

$$p_A + \rho_1 g h_1 = \rho_2 g h_2 + p_{atm}$$

$$p_A - p_{atm} = \rho_2 g h_2 - \rho_1 g h_1$$

$p_{gage}$

$$\rho_2 >> \rho_1$$

$$p_{A,gage} \approx \rho_2 g h_2$$





**Example 1**

Water flows in pipes A & B

SG(oil)=0.88

SG(Hg)=13.6

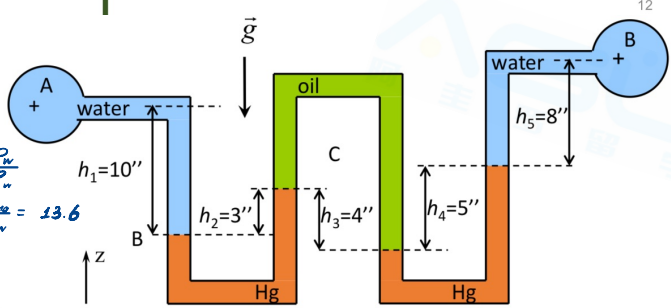
***Find:*** $\Delta p = p_A - p_B$

Specific gravity

$SG(H_2O) = 1$

$= \frac{\rho_w}{\rho_w}$

$SG(Hg) = \frac{\rho_{Hg}}{\rho_w} = 13.6$




downward: positive

upward: negative

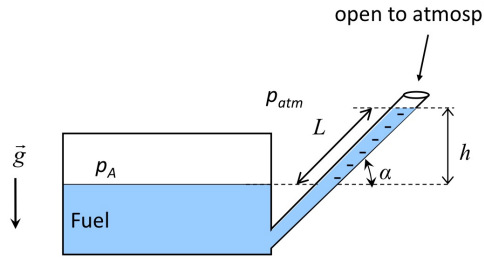
$$p_A + \rho_w g h_1 - \rho_{Hg} g h_2 + \rho_{oil} g h_3 - \rho_{Hg} g h_4 - \rho_w g h_5 = p_B$$

$$p_A - p_B = -\rho_w g h_1 + \rho_{Hg} g h_2 - \rho_{oil} g h_3 + \rho_{Hg} g h_4 + \rho_w g h_5$$

$$p_A - p_B = \rho_w g\left(-h_1 + \text{SG}_{Hg} h_2 - \text{SG}_{oil} h_3 + \text{SG}_{Hg} h_4 + h_5\right) =$$

$$998 \frac{\text{kg}}{\text{m}^3} 9.8 \frac{\text{m}}{\text{s}^2} (-10 + 13.6 \cdot 3 - 0.88 \cdot 4 + 13.6 \cdot 5 + 8) \text{ in} \times 0.0254 \frac{\text{m}}{\text{in}} = 25.7 \text{kPa}$$

**Example 2**

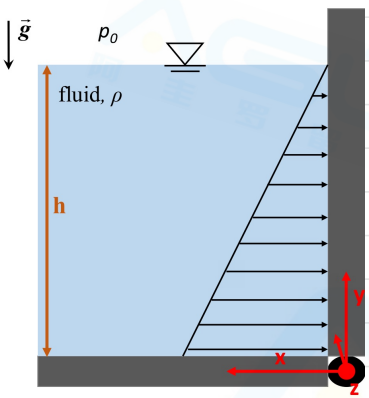



Reservoir tube inclined monometer

$$p_A - \rho g h = p_{\text{atm}}$$

$$\underbrace{h = L \sin\alpha}$$

$$p_{Agage} = \rho g L \sin\alpha$$




$$\frac{dP}{dy} = -\rho g$$

$$\int_{P_0}^{P} dP = -\rho g \int_{h}^{y} dy$$

$$P = P_0 + \rho g (h - y)$$

$$P_g = \rho g (h - y)$$

$$\vec{n} = +e_x$$

$$\vec{F}_R = \int d\vec{F}_R = \int -p\,d\vec{A} = \iint -p\vec{n}\,dA = \iint_A -p\,dA\,e_x$$

$$p_{page} = \rho g (h - y)$$

$$F_R = -\iint \rho g (h - y)\, dA\, e_x = -\rho g \iint_A (h - y)\, dA\, e_x$$

$$dA = dz\,dy = w\,dy$$

$$dz = w$$

$$F_R = -\rho g w \int_0^h (h - y)\, dy\, e_x = -\rho g w \cdot \frac{h^2}{2} e_x$$

**Torque** (vector)

$$\vec{M} = \vec{r} \times \vec{F}$$

$$d\vec{M} = \vec{r} \times d\vec{F}_R$$

$$d\vec{F}_R = -p\,d\vec{A}$$

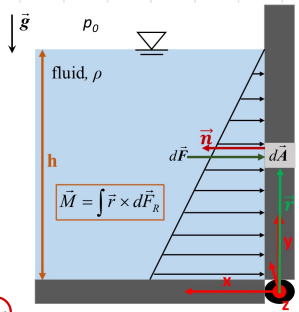
$$\vec{M} = \int \vec{r} \times d\vec{F}_R$$

$$= -\iint_A \vec{r} \times (p\,d\vec{A})$$

$$= \int_0^h \begin{vmatrix} e_x & e_y & e_z \\ x & y & z \\ -pw\,dy & 0 & 0 \end{vmatrix}$$

$$= \int_0^h e_z (x \cdot 0 - y \cdot (-pw\,dy))$$

$$= \int_0^h pyw\,dy \cdot (+e_z)$$




$$\vec{M} = \int_0^h pyw\,dy\,(+e_z)$$

Substituting the pressure distribution

$$p = \rho g = \rho g (h - y)$$

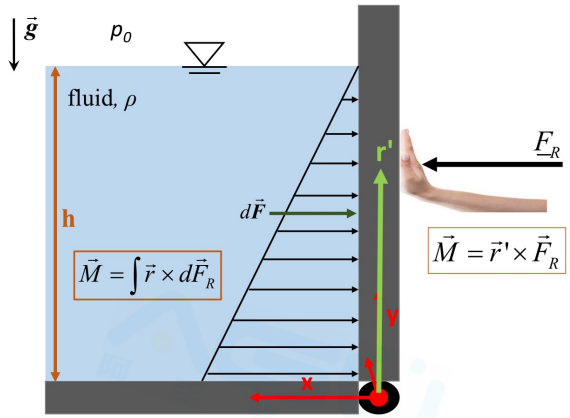
$$\vec{M} = \int_0^h \rho g (h - y) y w\,dy\,(+e_z)$$

$$\vec{M} = \rho g w \int_0^h (hy - y^2)\, dy\,(+e_z) =$$

$$= \rho g w \left[\left(\frac{hy^2}{2} - \frac{y^3}{3}\right)\right]_0^h (+e_z) =$$

$$\boxed{\vec{M} = \rho g w \frac{h^3}{6} (e_z)}$$

## Center of pressure

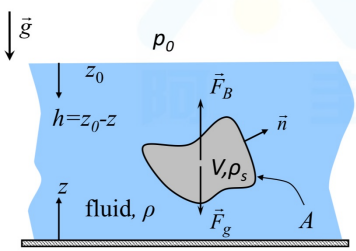



$$\vec{M} = \int \vec{r} \times d\vec{F}_R$$
$$\vec{M} = \vec{r}' \times \vec{F}_R$$
$$\int \vec{r} \times d\vec{F}_R = \vec{r}' \times \vec{F}_R$$
$$\rho g w \frac{h^3}{6} = y' \cdot \rho g w \frac{h^2}{2}$$
$$y' = \frac{h}{3}$$

The center of pressure: $r' = (\frac{h}{3}, \frac{w}{2})$

## Buoyancy Force 浮力 (Surface force)

- Any body completely or partially submerged in a fluid (gas or liquid) at rest is acted upon a force known as **buoyancy force:**




(Archimedes, 220 b.c.)
The net vertical pressure force on the object equals the force of gravity acting on the liquid displaced by the object

$$\vec{F}_B = -\oiint_A p\vec{n}\,dA$$
$$= -\oiint_A (p_0 + \rho g h)\vec{n}\,dA$$
$$= -\oiint_A [p_0 + \rho g(z_0 - z)h]\vec{n}\,dA =$$
$$= -\oiint_A \underbrace{[p_0 + \rho g z_0]}_{p_b = \text{Const}}\vec{n}\,dA + \oiint_A \rho g z \vec{n}\,dA$$



$$\vec{F}_B = -\oiint_A \underbrace{[p_0 + \rho g z_0]}_{p_b = \text{Const}}\vec{n}\,dA + \oiint_A \rho g z \vec{n}\,dA$$

- Now let's calculate both integrals:

1) $\oiint_A p_b \vec{n}\,dA = \iiint_V \nabla p_b\,dV = 0$

Gauss low

2) $\oiint_A \rho g z \vec{n}\,dA = \iiint_V \nabla(\rho g z)dV = \rho g V e_z$

For incompressible fluid $\nabla z = e_z$

$$\vec{F}_B = \rho g V e_z$$

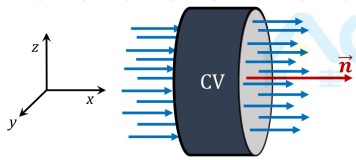
# Lecture 3

## Integral Balances - Mass Balance:

Average

control volume : cv

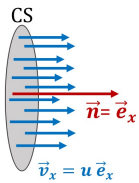
cross section : cs




Uniform flow filed (velocity):

Control volume (CV)

Cross section (CS)




### Uniform velocity:

**Volumetric flow rate:**

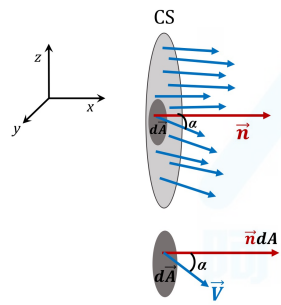
$$Q = \vec{V}_x \cdot \vec{A} = u\vec{e}_x \cdot A\vec{e}_x = uA \left[\frac{m}{sec} \cdot m^2\right] = \left[\frac{m^3}{sec}\right]$$

**Mass flow rate:**

$$\dot{m} = \rho \cdot \vec{V}_x \cdot \vec{A} = \rho \cdot u\vec{e}_x \cdot A\vec{e}_x = \rho uA \left[\frac{kg}{m^3} \cdot \frac{m^3}{sec}\right] = \left[\frac{Kg}{sec}\right]$$

### Non-uniform velocity:

Non - Uniform flow filed (velocity):

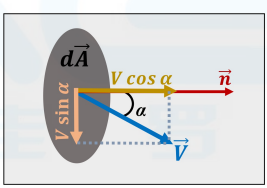



Control volume (CV), Cross section (CS)

**Mass flow rate through A :**

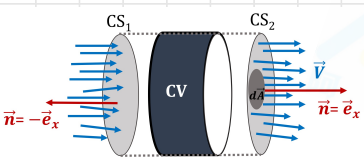
$$\dot{m} = \rho \iint_{cs} \vec{V} \cdot d\vec{A} = \rho \iint_{cs} V \cos\alpha \, dA$$

**Mass flow rate through $dA$:**




### Mass Balance:

$$\frac{\partial}{\partial t}\iiint_{cv} \rho dV + \oiint_{c.s} \rho\vec{V} \cdot d\vec{A} = 0$$

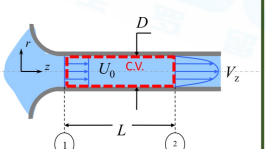



**accumulation (CV) + [Inlet ($CS_1$) – Outlet($CS_2$)] = 0**

### 3.1 Integral Balances – Mass Balance - EXAMPLE 1

A. Find the maximum velocity $U_{max}$ at section 2 (in terms of $U_0$);

$$\frac{V_z(r)}{U_{max}} = 1 - \left(\frac{r}{R}\right)^2$$




Solution

- First we should define control volume (CV)
- Integral mass balance over the CV:

Assumptions:

- Steady state (no change with time)
- Incompressible fluid - $\rho = const$

$$\frac{\partial}{\partial t}\iiint_{V.} \rho dV + \oiint_{C.S.} \rho\vec{V} \cdot d\vec{A} = 0$$

Steady state

$$\oiint_{C.S.} \rho\vec{V} \cdot \vec{n}\, dA = 0$$

$$\iint_{A_1} U_0\vec{e}_z \cdot (-\vec{e}_z)\, dA + \iint_{A_2} V_z(r) \cdot \vec{e}_z \cdot \vec{e}_z\, d\vec{A} = 0$$

$$-U_0\pi R^2 + \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} U_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] r\, d\theta dr = 0$$

r ← Jacobian

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$-U_0\pi R^2 + \int_0^R U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r\, dr = 0$$

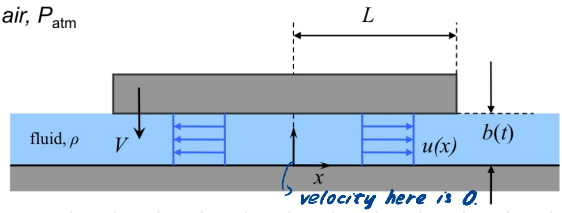
$$-U_0\pi R^2 + U_{max}\left(\pi r^2 - \frac{\pi r^4}{2R^2}\right)\Big|_{r=0}^{r=R} = 0$$

$$-U_0\pi R^2 + 2\pi U_{max}\frac{R^2}{4} = 0 \Rightarrow U_{max} = 2U_0$$

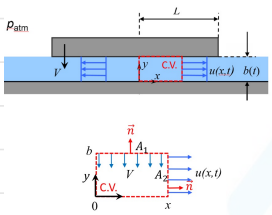
# 3.1 Integral Balances – Mass Balance - EXAMPLE 2

❑**(Final exam 2008B):** Layer of incompressible fluid of density $\rho$ is sandwiched between two parallel plates as shown in the figure. The lower plate is stationary, while the upper plate is forced towards the lower plate at constant velocity $V$. The length of the upper plate is $2L$ and the width (normal to the board) is $W$, so that $W>>L$. As a result, the fluid is squeezed out of the gap between the plates with velocity $u(x)$, that can be assumed uniform at any transverse cross-section (see the figure). The viscosity can be neglected.

**A. For a proper control volume find $u(x)$ as a function of $V$, $x$ and $b(t)$**




A. For a proper control volume find $u(x)$ as a function of $V$, $x$ and $b(t)$




**Solution**

- First we should define control volume (CV)
- Integral mass balance over the CV:

Assumptions:

- No accumulation or mass loss in the CV
- Incompressible fluid - $\rho = const$

$$\frac{\partial}{\partial t}\iiint_{C.V.}\rho dV + \oiint_{C.S.}\rho\vec{V}\cdot d\vec{A} = 0$$

$$\rho\oiint_{C.S.}\vec{V}\cdot\vec{n}\,dA = 0 \qquad \rho = const$$

$$b(t) = b_0 - Vt$$

$$u(y) = -V\vec{e_y}$$

$$u(x,t) = u(x,t)\vec{e_x}$$

$$\int_0^x\int_0^W -V\vec{e_y}\cdot(\vec{e_y})\,dz\,dx + \int_0^{b(t)}\int_0^W u(x,t)\vec{e_x}\cdot(\vec{e_x})\,dz\,dy = 0$$

$$-VxW + u(x,t)W\,b_0(t) = 0$$

$$-VxW + u(x,t)W(b_0 - Vt) = 0$$

$$u(x,t) = \frac{Vx}{b_0 - Vt}$$

**Linear Momentum Balance**

$$\Sigma\vec{F} = \left.\frac{d(m\vec{v})}{dt}\right|_{sys} = m\vec{a}$$

$$\Sigma\vec{F} = \oiint_{c.s}\rho\vec{V}(\vec{V}\cdot\vec{n})\,dA + \frac{\partial}{\partial t}\iiint_{c.v}\rho\vec{V}\,dV$$

$$\Sigma\vec{F} = \frac{d}{dt}\iiint_{sys}\vec{V}\rho dV \overset{\text{雷诺运输}}{=} \frac{\partial}{\partial t}\iiint_{cv}\vec{V}\rho dV + \iint_{cs}\vec{V}\rho(\vec{V}\cdot\vec{n})\,dA$$

体系动量随时间变化　　控制体内动量随时间变化　　净流出控制体动量

$m = \rho V$ (volume)

$\dot{m} = \rho\cdot\vec{A}\cdot\vec{v}$ (velocity)

**Or in a scalar (component) form:**

$$\sum F_x = \oiint_{C.S.}\rho V_x(\vec{V}\cdot\vec{n})dA + \frac{\partial}{\partial t}\iiint_{C.V.}\rho V_x dV$$

$$\sum F_y = \oiint_{C.S.}\rho V_y(\vec{V}\cdot\vec{n})dA + \frac{\partial}{\partial t}\iiint_{C.V.}\rho V_y dV$$

$$\sum F_z = \oiint_{C.S.}\rho V_z(\vec{V}\cdot\vec{n})dA + \frac{\partial}{\partial t}\iiint_{C.V.}\rho V_z dV$$