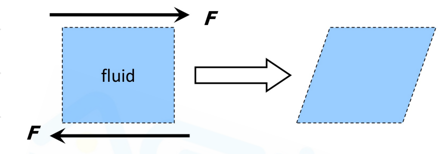
Lecture 1

Fluid: A substance in the Liquid or Gas phase is refer to be fluid.

Mechanics: is the branch of physical science that deals with the behaviour of stationary and moving bodies when subjected to force.

Fluids:

Def 1. Fluid is a substance that continuously deforms (i.e. flows) under an applied shear stress.



Def 2. Substances that are unable to support shear stresses in static equilibrium and possess the ability to flow.

Def 3. According to Rheology there is no clear distinction between fluid and solid. Fluidity → Debora number: $D_e = \frac{t_{rel}}{t_{exp}}$

t_{rel} → Relaxation time

t_{exp} → Time of experiment

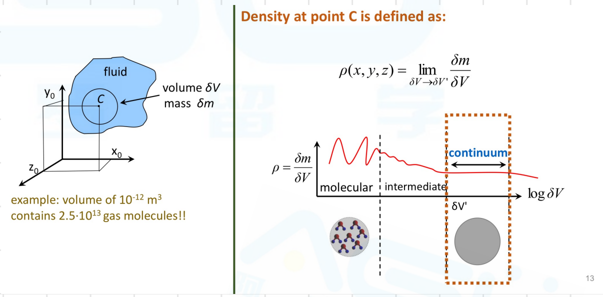
$D_e \to \infty$ Solid-like

$D_e \approx 1$ Intermediate

$D_e \approx 0$ Fluid-like

The Concept of "Continuum".

The material is treated as continuum, meaning that it can be continually sub-divided into infinitesimal elements with properties (such as density, pressure, temp. etc) being those of the bulk material.



Fluid Velocity

$\vec{V} = \vec{V}(x,y,z,t) = V_x \underset{\hat{i}}{e_x} + V_y \underset{\hat{j}}{e_y} + V_z \underset{\hat{k}}{e_z}$

$V = |\vec{V}| = (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}}$

Steady-State (Velocity is constant): $\vec{V} = \vec{V}(x,y,z)$

$\frac{\partial \vec{V}}{\partial t} = 0$

Plane Flow: $\vec{V} = \vec{V}(x,y,t) = V_x(x,y,t)\,\hat{i} + V_y(x,y,t)\,\hat{j}$

Unidirectional Flow: $\vec{V} = V_x(y,z)\,\hat{i}$

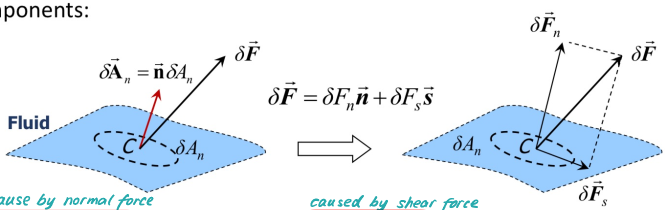
Pressure and Stress

Body force: acting (inside the item) throughout the volume of fluid element. Care about Mass (example: gravity)

Surface force: internal forces acting across surfaces of fluid elements. Can be decomposed:

Pressure (normal), Shear (tangential) Care about Volume.

We can decompose the **surface force** acting on a surface of fluid element at point *C* into **normal** and **tangential** components:



cause by normal force　　　　　　caused by shear force

We can then define normal stress (σ) and tangential or shear stress (τ) as:

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n}, \quad \tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_s}{\delta A_n}$$

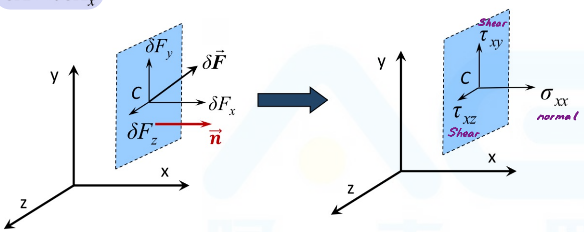
Stresses exerted on $\delta \vec{A}_n = \vec{n}\,\delta A_n$

Due to this definition, σ and τ depend on choice of the surface orientation!

We look for an alternative definition of stress exerted by the fluid at point *C* that does not depend on the choice of surface passing though *C* :

For Example we choose plane with normal at x direction:

$\delta \vec{A} = \vec{i}\,\delta A_x$



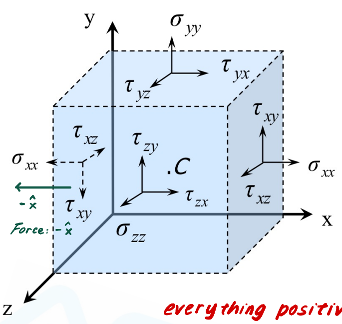
$$\boxed{\delta \vec{F} = \delta F_x \vec{i} + \delta F_y \vec{j} + \delta F_z \vec{k}}$$

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x}$$

$$\tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x}$$

$$\tau_{xz} = \lim_{\delta A_x \to 0} \frac{\delta F_z}{\delta A_x}$$

double index:
1st: surface orientation
2nd: force direction

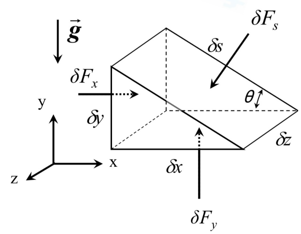


everything positive.

All stress components here are positive!

Reversing the direction of either normal or force: revert sign

Flipping both: preserves it!



Static Fluid: Shear Stress 缺席 absent.

Normal Stress: $\Sigma \vec{F} = 0$

$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$ (Pascal Law)

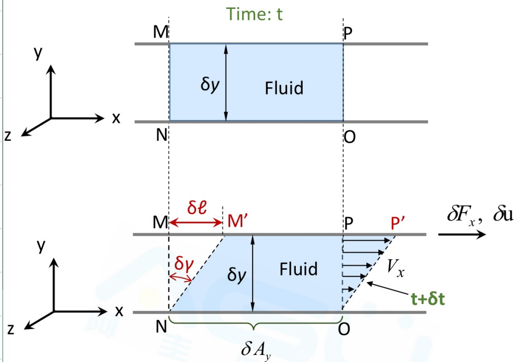
$$\boxed{P = -\frac{1}{3}\sum_{i=1}^{3} \sigma_{ii} = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)}$$

(Pressure uniform everywhere.)

三个名字

Viscosity → Shear Stress, Shear/Strain/Deformation Rate

For a fluid encapsulated between two long parallel plates:



$$\frac{\delta\gamma}{\delta t} = \frac{\delta u}{\delta y} = \dot{\gamma} = \text{Const}$$

Shear stress:

direction: $\hat{y}$

force: $\hat{x}$

$$\tau = \tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y}$$

Strain(shear) rate:

Angle

$$\lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \frac{d\gamma}{dt} = \dot{\gamma}$$

δF_x, δu

We can't describe the change with time, thus we use measurable parameters:

$$\underbrace{\delta\ell \approx \delta u \delta t; \quad \delta\ell = \delta y \tan\delta\gamma \approx \delta y \delta\gamma}$$

$$\frac{\delta\gamma}{\delta t} = \frac{\delta u}{\delta y} = \dot{\gamma} = \text{Const}$$

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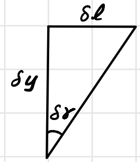


$\delta\ell = \delta u \cdot \delta t$

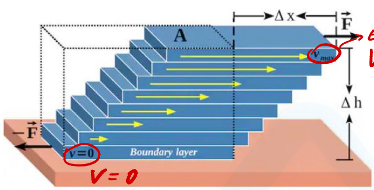
$\frac{\delta\ell}{\delta y} = \tan\delta\gamma$

if it's very small then $\tan\delta\gamma = \delta\gamma$

$\delta\ell = \delta y \cdot \delta\gamma$

$\delta y \cdot \delta\gamma = \delta u \cdot \delta t$

$\frac{\delta\gamma}{\delta t} = \frac{\delta u}{\delta y} = \dot{\gamma}$



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Exactly the velocity of the surface

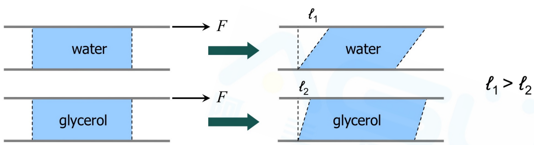
V_{max}

Shear Stress exists on each layer.

In any unidirectional flow

$$\boxed{\frac{d\gamma}{dt} = \frac{dV_x}{dy}}$$

- Fluids for which shear stress $\tau = \tau_{yx}$ is linearly proportional to strain (shear) rate $\dot{\gamma}$ are called **Newtonian fluids**, e.g., water, gases, oils, alcohols, naphtha, etc. and the proportion constant is the **viscosity.**



We then say that glycerol is more viscous than water!

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Viscosity - Newtonian Fluid

Shear stress ⟺ Strain rate

$\tau_{yx} \iff \dot{\gamma}$

Newtonian Fluid (proportionality coefficient μ)

$\tau = \mu\dot{\gamma}$

viscosity (How much material will resist the force.)

if $\mu \to \infty$, solid-like

Dimensions and units of viscosity:

$$\tau = \left[\frac{F}{L^2}\right], \dot{\gamma} = \left[\frac{1}{T}\right] \Rightarrow \mu = \left[\frac{F \cdot T}{L^2}\right] = \left[\frac{M}{L \cdot T}\right]$$

In the SI system we use the units of Pa·s,

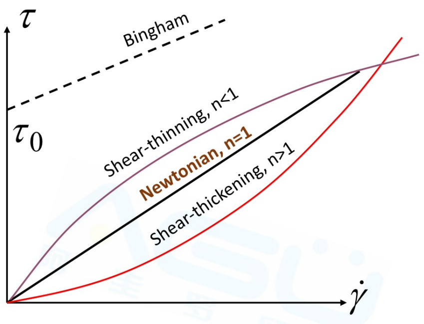
The viscosity of water at room temp.: 0.01P = 1cP = 10^{-3} Pa ·s (1 P=0.1 Pa·s)

Kinematic viscosity: $\nu = \frac{\mu}{\rho} = \left[\frac{M}{L \cdot T}\frac{L^3}{M}\right] = \frac{L^2}{T}$

Kinematic Viscosity $\nu = \frac{\mu}{\rho} = \left[\frac{M}{L \cdot T} \cdot \frac{L^3}{M} = \frac{L^2}{T}\right]$

Viscosity – non-Newtonian Fluid 非牛顿流体

- Fluids for which the above does not hold are called **non-Newtonian fluids**, e.g., polymer solutions, emulsions, dispersions, gels, etc.



- Generalized Newtonian fluid:

$$\boxed{\tau = m\dot{\gamma}^n = \underbrace{m|\dot{\gamma}|^{n-1}}_{\mu_{app}(\dot{\gamma})}\dot{\gamma}}$$

$n<1: \dot{\gamma}\uparrow \ \mu_{app}\downarrow$ - shear - thinning 剪切稀化

$n>1: \dot{\gamma}\uparrow \ \mu_{app}\uparrow$ - shear - thickening

- Bingham fluid: $\boxed{\tau = \tau_0 + \mu\dot{\gamma}}$

- Viscoelastic fluid: $\boxed{\tau = G\gamma + \mu\dot{\gamma}}$

(Kelvin model)

2024.03.07

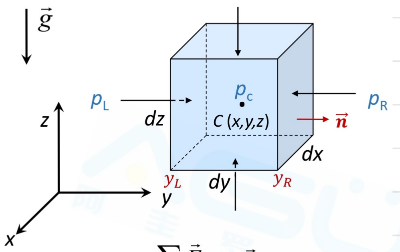
Lecture 2

(normal force)

Static fluid: shear stress absent; the only surface force is due to pressure.
(or fluid in a rigid-body motion)

In a static fluid: $\Sigma \vec{F} = 0$

In a fluid in a rigid-body motion: $\Sigma \vec{F} = m\vec{a}$



$$\sum \vec{F} = m\vec{a}$$

density

Body Force: $d\vec{F}_g = \vec{g}\,dm = \vec{g}\rho\,dV = \vec{g}\rho\,dx\,dy\,dz$

Force Balance (y-direction)

$P_R = P_c$

$d\vec{F}_p = -p\,d\vec{A}$

$d\vec{F}_{p,R} = -p_R\,d\vec{A}$

$d\vec{F}_{p,L} = -p_L\,d\vec{A}$

Assuming we know what is P_c, if we expand the pressure at y_R and y_L using Tylor series:

$$p_R = p_c + \left(\frac{\partial p}{\partial y}\right)_c (y_R - y) + \cdots = p_c + \left(\frac{\partial p}{\partial y}\right)_c \frac{dy}{2} + \cdots$$

省略

$$p_L = p_c + \left(\frac{\partial p}{\partial y}\right)_c \underbrace{(y_L - y)}_{\text{minus}} + \cdots = p_c - \left(\frac{\partial p}{\partial y}\right)_c \frac{dy}{2} + \cdots$$

省略

$\Sigma \vec{F} = d\vec{F}_{p,R} + d\vec{F}_{p,L}$

$$d\vec{F}_s = -\underbrace{\left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)_c (dx\,dz)(-\vec{j})}_{d\vec{F}_{p,L}} + \underbrace{\left(p + \frac{\partial p}{\partial y}\cdot\frac{dy}{2}\right)_c (dx\,dz)(\vec{j})}_{d\vec{F}_{p,R}} \;(-1)$$

P_c $\quad$ P_c

$$= -\left(\frac{\partial p}{\partial y}\right)_c dx\,dy\,dz\,\vec{j}$$

Fluid static equation

$$-\nabla p + \rho(\vec{g} - \vec{a}) = 0$$

Victorian equation

$$-\nabla P + \rho g = 0$$

$$\vec{\nabla} = \frac{\partial}{\partial x}e_x + \frac{\partial}{\partial y}e_y + \frac{\partial}{\partial z}e_z$$

$$\vec{\nabla}P = \frac{\partial P}{\partial x}e_x + \frac{\partial P}{\partial y}e_y + \frac{\partial P}{\partial z}e_z$$

$$d\vec{F}_s = -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right)dx\,dy\,dz = -(\nabla P)_c\,dV$$

$\Sigma \vec{F} = m\vec{a}$

$$d\vec{F} = d\vec{F}_s + d\vec{F}_g = (-\nabla P + \rho\vec{g})\,dV = \vec{a}\rho\,dV$$

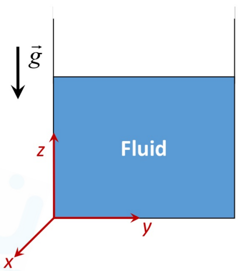
$$\frac{d\vec{F}}{dV} = \vec{a}\rho = -\nabla P + \rho\vec{g}$$

$$-\frac{\partial P}{\partial x} + \rho g_x = 0$$

$$-\frac{\partial P}{\partial y} + \rho g_y = 0$$

$$-\frac{\partial P}{\partial z} + \rho g_z = 0$$

Basic application of the Fluid statics equation:



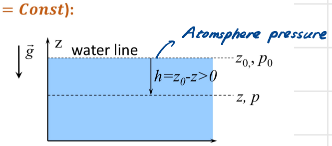
$\vec{g}_z = -\vec{g}$

$$\underbrace{-\frac{\partial p}{\partial x} = 0, \quad -\frac{\partial p}{\partial y} = 0,}_{p = p(z)} \quad -\frac{\partial p}{\partial z} - \rho g = 0 \quad (g = |\vec{g}|)$$

$$\boxed{\frac{dp}{dz} = -\rho g}$$

In a static fluid (or in a steady rigid-body motion) the pressure changes (decreases) only with elevation (z)!

Pressure change in an incompressible fluid ($\rho = Const$):



$$-\frac{\partial P}{\partial z} \boxed{+\rho g_z} = 0$$

$$-\frac{\partial P}{\partial z} \boxed{-\rho g} = 0$$

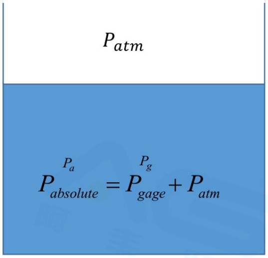
$$\frac{dP}{dz} = -\rho g$$

$$\int_{P_0}^{P} dP = -\rho g \int_{z_0}^{z} dz$$

$$P - P_0 = -\rho g \underbrace{(z - z_0)}_{-h}$$

$$P = P_0 + \rho g h$$

- **Absolute pressure is measured relative to a perfect vacuum, whereas gage pressure is measured relative to the atmospheric pressure.**
- P_{gage} is the Pressure measured relative to a reference pressure. The reference pressure is arbitrary. Usually it is taken to be the atmospheric pressure.

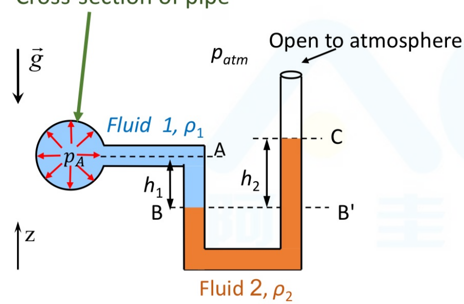


$$\overset{P_a}{P_{absolute}} = \overset{P_g}{P_{gage}} + P_{atm} \quad \Longleftrightarrow \quad p_{gage} = p_{abs} - p_{atm}\,,$$

$$p_{atm} = 101.3 \text{ kPa, } 760 \text{ mmHg}$$

Pressure measurement by manometers is based on:

$$\boxed{p = p_{atm} + \rho g h}$$



$$p_B = p_A + \rho_1 g h_1$$

$$p_{B'} = \rho_2 g h_2 + p_c$$

$$p_c = p_{\text{atm}}$$

$$p_B = p_{B'}$$

$$p_A + \rho_1 g h_1 = \rho_2 g h_2 + p_{atm}$$

$$\underbrace{p_A - p_{atm}}_{P_{gage}} = \rho_2 g h_2 - \rho_1 g h_1$$

$$\rho_2 >> \rho_1$$

$$\boxed{p_{A,gage} \approx \rho_2 g h_2}$$

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Example 1

Water flows in pipes A & B

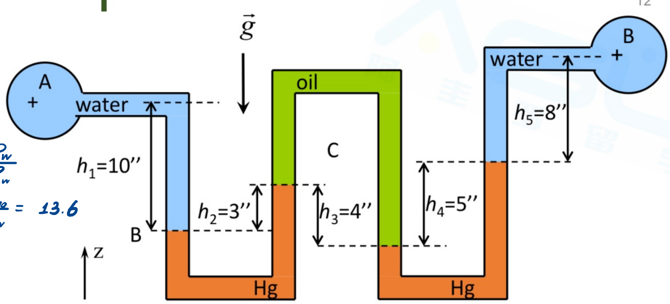
SG(oil)=0.88 Specific gravity

$SG(H_2O) = 1 = \frac{\rho_w}{\rho_w}$

SG(Hg)=13.6

$SG(Hg) = \frac{\rho_{Hg}}{\rho_w} = 13.6$

Find: $\Delta p = p_A - p_B$



downward: positive

$$p_A + \rho_w g h_1 - \rho_{Hg} g h_2 + \rho_{oil} g h_3 - \rho_{Hg} g h_4 - \rho_w g h_5 = p_B$$

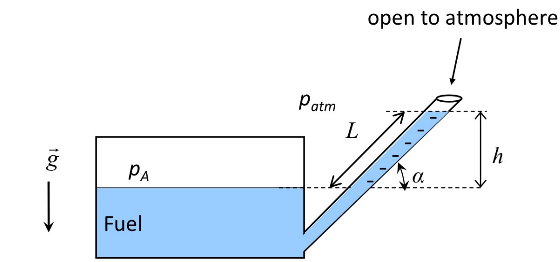
upward: negative

$$p_A - p_B = -\rho_w g h_1 + \rho_{Hg} g h_2 - \rho_{oil} g h_3 + \rho_{Hg} g h_4 + \rho_w g h_5$$

$$p_A - p_B = \rho_w g\left(-h_1 + \mathrm{SG}_{Hg} h_2 - \mathrm{SG}_{oil} h_3 + \mathrm{SG}_{Hg} h_4 + h_5\right) =$$

$$998 \frac{\text{kg}}{\text{m}^3} 9.8 \frac{\text{m}}{\text{s}^2} (-10 + 13.6 \cdot 3 - 0.88 \cdot 4 + 13.6 \cdot 5 + 8) \text{ in} \times 0.0254 \frac{\text{m}}{\text{in}} = 25.7\text{kPa}$$

Example 2

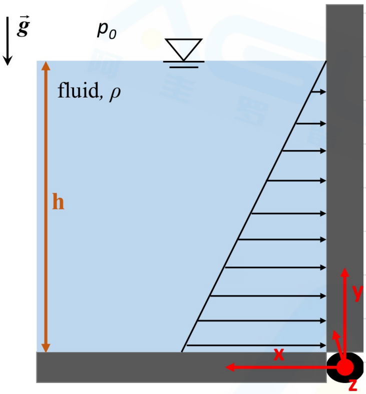


Reservoir tube inclined monometer

$$p_A - \rho g h = p_{\text{atm}}$$

$$\underbrace{h = L\sin\alpha}$$

$$p_{Agage} = \rho g L \sin\alpha$$



$$\frac{dP}{dy} = -\rho g$$

$$\int_{P_0}^{P} dP = -\rho g \int_h^y dy$$

$$P = P_0 + \rho g (h-y)$$

$$P_g = \rho g (h-y)$$

$$\vec{n} = +e_x$$

$$\vec{F}_R = \int d\vec{F}_R = \int -p\,d\vec{A} = \iint -p\vec{n}\,dA = \iint_A -p\,dA\,e_x$$

$$P_{page} = \rho g (h-y)$$

$$F_R = -\iint \boxed{\rho g (h-y)}\,dA\,e_x = -\rho g \iint_A (h-y)\,dA\,e_x$$

$$dA = dz\,dy = w\,dy$$

$$dz = w$$

$$F_R = -\rho g w \int_0^h (h-y)\,dy\,e_x = -\rho g w \cdot \frac{h^2}{2}\,e_x$$

Torque (vector)

$$\vec{M} = \vec{r} \times \vec{F}$$

$$d\vec{M} = \vec{r} \times d\vec{F}_R$$

$$d\vec{F}_R = -p\,d\vec{A}$$

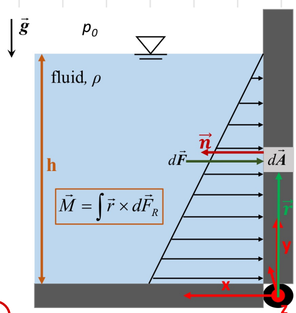
$$\vec{M} = \int \vec{r} \times d\vec{F}_R$$

$$= -\iint_A \vec{r} \times (p\,d\vec{A})$$

$$= \int_0^h \begin{vmatrix} e_x & e_y & e_z \\ x & y & z \\ -pw\,dy & 0 & 0 \end{vmatrix}$$

$$= \int_0^h e_z\,(x \cdot 0 - y \cdot (-pw\,dy))$$

$$= \int_0^h p y w\,dy \cdot (+e_z)$$



$$\vec{M} = \int_0^h p y w dy (+e_z)$$

Substituting the pressure distribution

$$p = \cancel{p_g} = \rho g (h - y)$$

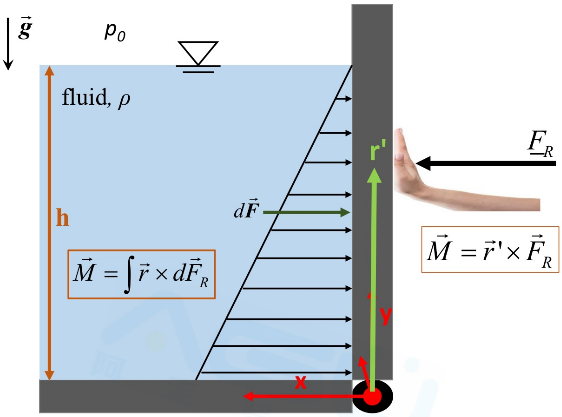
$$\vec{M} = \int_0^h \rho g (h - y) y w dy (+e_z)$$

$$\vec{M} = \rho g w \int_0^h (hy - y^2)\,dy (+e_z) =$$

$$= \rho g w \left[\left(\frac{hy^2}{2} - \frac{y^3}{3}\right)\right]_0^h (+e_z) =$$

$$\boxed{\vec{M} = \rho g w \frac{h^3}{6} (e_z)}$$

Center of pressure



$$\vec{M} = \int \vec{r} \times d\vec{F}_R$$

$$\vec{M} = \vec{r}' \times \vec{F}_R$$

$$\int \vec{r} \times d\vec{F}_R = \vec{r}' \times \vec{F}_R$$

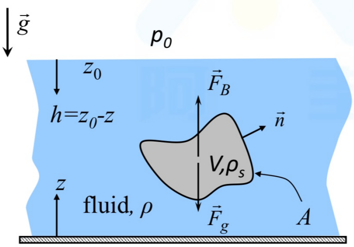
$$\rho g w \frac{h^3}{6} = y' \cdot \rho g w \frac{h^2}{2}$$

$$y' = \frac{h}{3}$$

The center of pressure: $r' = (\frac{h}{3}, \frac{w}{2})$

Buoyancy Force 浮力 (Surface force)

- Any body completely or partially submerged in a fluid (gas or liquid) at rest is acted upon a force known as **buoyancy force:**



(Archimedes, 220 b.c.)
The net vertical pressure force on the object equals the force of gravity acting on the liquid displaced by the object

$$\vec{F}_B = -\oint_A p\vec{n}\,dA$$

$$= -\oint_A (p_0 + \rho g h)\vec{n}\,dA$$

封闭

$$= -\oint_A [p_0 + \rho g(z_0 - z)h]\vec{n}\,dA =$$

$$= -\oint_A \underbrace{[\boldsymbol{p_0 + \rho g z_0}]}_{p_b\,=\,\text{Const}}\boldsymbol{\vec{n}\,dA} + \oint_A \boldsymbol{\rho g z \vec{n}\,dA}$$

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$$\vec{\boldsymbol{F}}_B = -\oint_A \underbrace{[\boldsymbol{p_0 + \rho g z_0}]}_{p_b\,=\,\text{Const}}\boldsymbol{\vec{n}\,dA} + \oint_A \boldsymbol{\rho g z \vec{n}\,dA}$$

- Now let's calculate both integrals:

1) $\oint_A p_b \vec{n}\,dA = \underbrace{\iiint_V}_{\text{Gauss low}} \nabla p_b\,dV = 0$

2) $\oint_A \rho g z \vec{n}\,dA = \iiint_V \nabla(\rho g z)dV = \rho g V e_z$

For incomprissable fluid $\nabla z = e_z$

$$\vec{\boldsymbol{F}}_B = \boldsymbol{\rho g V e_z}$$

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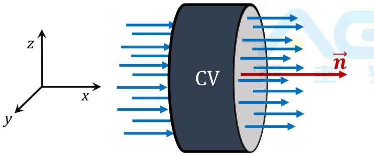
Lecture 3

Integral Balances - Average Mass Balance:

control volume : cv

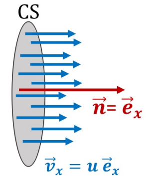
cross section : cs



Uniform flow filed (velocity):

Control volume (CV)

Cross section (CS)



Uniform velocity:

Volumetric flow rate:

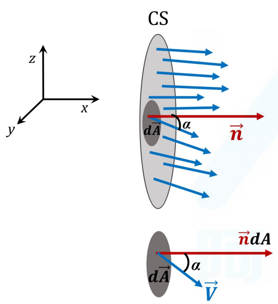
$$Q = \vec{V}_x \cdot \vec{A} = u\vec{e}_x \cdot A\vec{e}_x = uA \left[\frac{m}{sec}\cdot m^2\right] = \left[\frac{m^3}{sec}\right]$$

Mass flow rate:

$$\dot{m} = \rho \cdot \vec{V}_x \cdot \vec{A} = \rho \cdot u\vec{e}_x \cdot A\vec{e}_x = \rho u A \left[\frac{kg}{m^3}\cdot\frac{m^3}{sec}\right] = \left[\frac{kg}{sec}\right]$$

Non-uniform velocity:

Non - Uniform flow filed (velocity):

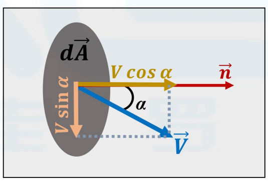


Control volume (CV), Cross section (CS)

Mass flow rate through A:

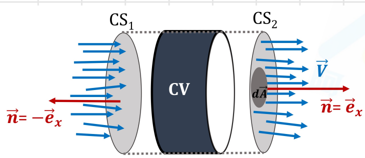
$$\dot{m} = \rho \iint_{cs} \vec{V}\cdot d\vec{A} = \rho \iint_{cs} V\cos\alpha \, dA$$

Mass flow rate through dA:



Mass Balance:

$$\frac{\partial}{\partial t}\iiint_{c.v} \rho \, dV + \oiint_{c.s} \rho \vec{V}\cdot d\vec{A} = 0$$

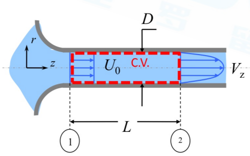


accumulation (CV) + [Inlet (CS_1) – Outlet(CS_2)] = 0

3.1 Integral Balances – Mass Balance - EXAMPLE 1

A. Find the maximum velocity U_{max} at section 2 (in terms of U_0);

$$\frac{V_z(r)}{U_{max}} = 1 - \left(\frac{r}{R}\right)^2$$



Solution

- First we should define control volume (CV)
- Integral mass balance over the CV:

Assumptions:

- Steady state (no change with time)
- Incompressible fluid - $\rho = const$

$$\cancel{\frac{\partial}{\partial t}\iiint_{C.V.} \rho \, dV} + \oiint_{C.S.} \rho \vec{v}\cdot d\vec{A} = 0$$

Steady state

$$\oiint_{C.S.} \rho \vec{V}\cdot\vec{n}\, dA = 0$$

$$\iint_{A_1} U_0 \vec{e}_z \cdot (-\vec{e}_z)\, dA + \iint_{A_2} V_z(r)\, \vec{e}_z \cdot \vec{e}_z \, dA = 0$$

$$-U_0 \pi R^2 + \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} U_{max}\left[1-\left(\frac{r}{R}\right)^2\right] r \, d\theta \, dr = 0$$

Jacobian

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$-U_0 \pi R^2 + \int_0^R U_{max}\left(1-\left(\frac{r}{R}\right)^2\right) 2\pi r \, dr = 0$$

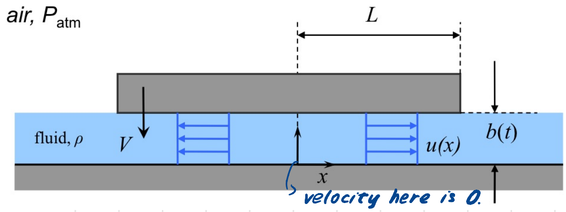
$$-U_0 \pi R^2 + U_{max}\left(\pi r^2 - \frac{\pi r^4}{2R^2}\right)\Bigg|_{r=0}^{r=R} = 0$$

$$-U_0 \pi R^2 + 2\pi U_{max} \frac{R^2}{4} = 0 \Rightarrow U_{max} = 2U_0$$

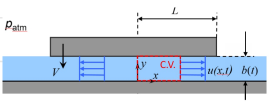
3.1 Integral Balances – Mass Balance - EXAMPLE 2

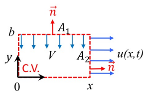
❑**(Final exam 2008B):** Layer of incompressible fluid of density ρ is sandwiched between two parallel plates as shown in the figure. The lower plate is stationary, while the upper plate is forced towards the lower plate at constant velocity V. The length of the upper plate is $2L$ and the width (normal to the board) is W, so that W>>L. As a result, the fluid is squeezed out of the gap between the plates with velocity $u(x)$, that can be assumed uniform at any transverse cross-section (see the figure). The viscosity can be neglected.

A. For a proper control volume find $u(x)$ as a function of V, x and $b(t)$



A. For a proper control volume find $u(x)$ as a function of V, x and $b(t)$





Solution

- **First we should define control volume (CV)**
- **Integral mass balance over the CV:**

Assumptions:

- No accumulation or mass loss in the CV
- Incompressible fluid - $\rho = const$

$$\boxed{\cancel{\frac{\partial}{\partial t}\iiint_{C.V.}\rho dV} + \oiint_{C.S.}\rho\vec{V}\cdot d\vec{A} = 0}$$

$$\boxed{\cancel{\rho}\oiint_{C.S.}\vec{V}\cdot\vec{n}\, dA = 0} \quad \rho = const$$

$$b(t) = b_0 - vt$$

$$u(y) = -v\vec{e}_y$$

$$u(x,t) = u(x,t)\vec{e}_x$$

$$\iint_0^{x\ \ w} -v\vec{e}_y\cdot(\vec{e}_y)\,dz\,dx + \iint_0^{b(t)\ \ w} u(x,t)\vec{e}_x\cdot(\vec{e}_x)\,dz\,dy = 0$$

$$-Vxw + u(x,t)\,w\,b_0(t) = 0$$

$$-Vxw + u(x,t)\,w\,(b_0 - vt) = 0$$

$$u(x,t) = \frac{Vx}{b_0 - vt}$$

$m = \rho V$ (V → volume)

$\dot{m} = \rho\cdot\vec{A}\cdot\vec{v}$ ($\vec{v}$ → velocity)

Linear Momentum Balance

$$\Sigma\vec{F} = \left.\frac{d(m\vec{v})}{dt}\right|_{sys} = m\vec{a}$$

$$\Sigma\vec{F} = \oiint_{c.s}\rho\vec{v}(\vec{v}\cdot\vec{n})\,dA + \frac{\partial}{\partial t}\iiint_{c.v}\rho\vec{v}\,dv$$

$$\Sigma\vec{F} = \underbrace{\frac{d}{dt}\iiint_{sys}\vec{v}\rho\,dV}_{\text{体系动量随时间变化}} \overset{\text{雷诺运输}}{=} \underbrace{\frac{\partial}{\partial t}\iiint_{cv}\vec{v}\rho\,dV}_{\text{控制体内动量随时间变化}} + \underbrace{\iint_{cs}\vec{v}\rho(\vec{v}\cdot\vec{n})\,dA}_{\text{净流出控制体动量}}$$

Or in a scalar (component) form:

$$\sum F_x = \oiint_{C.S.}\rho V_x(\vec{V}\cdot\vec{n})dA + \frac{\partial}{\partial t}\iiint_{C.V.}\rho V_x dV$$

$$\sum F_y = \oiint_{C.S.}\rho V_y(\vec{V}\cdot\vec{n})dA + \frac{\partial}{\partial t}\iiint_{C.V.}\rho V_y dV$$

$$\sum F_z = \oiint_{C.S.}\rho V_z(\vec{V}\cdot\vec{n})dA + \frac{\partial}{\partial t}\iiint_{C.V.}\rho V_z dV$$