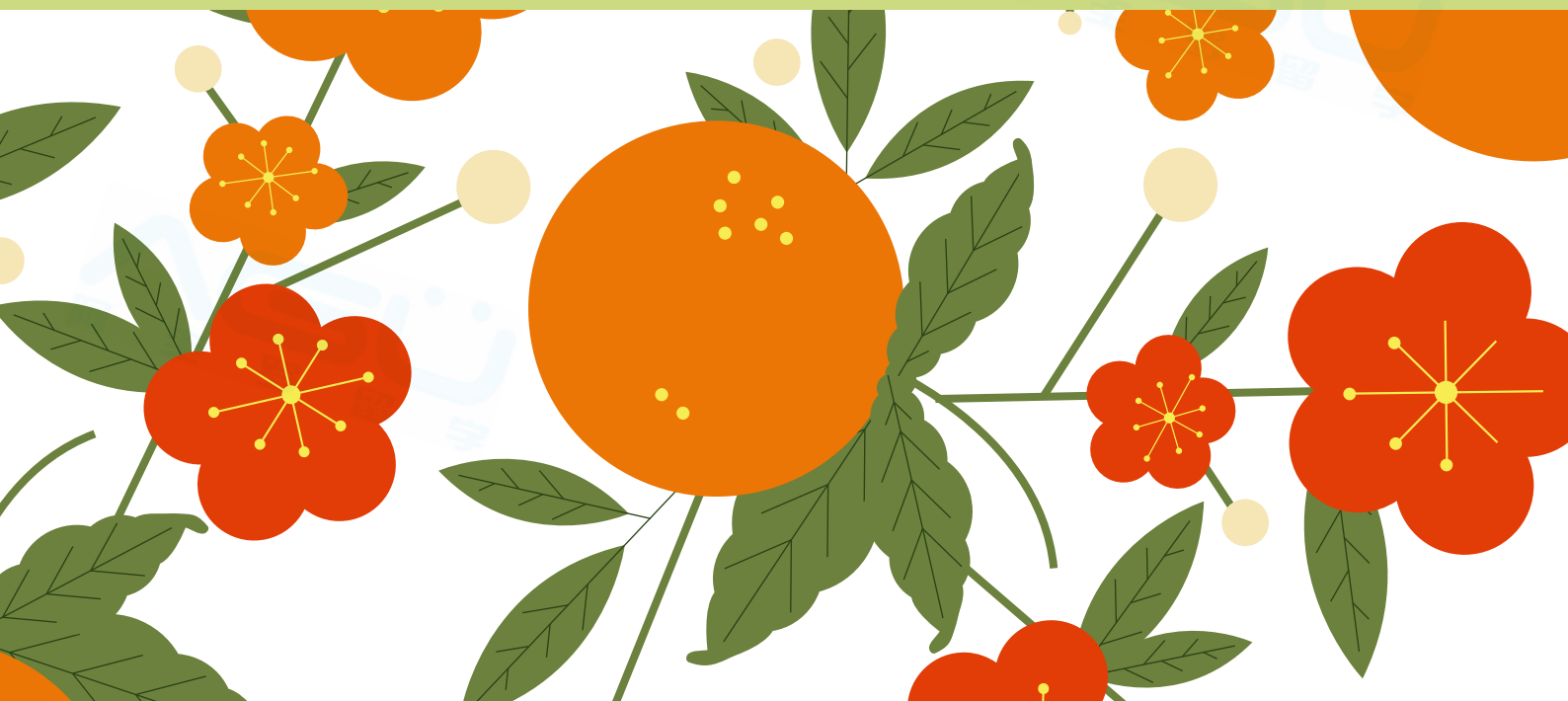




office hour: 13:30-15:30 ^(原计划) L210 Monday

Real numbers, Intervals and Inductions

Counting numbers: natural numbers.	$\mathbb{N}$	$\{1, 2, 3, 4, \dots\}$	$\mathbb{N} \subseteq \mathbb{Z}$
integers	$\mathbb{Z}$	$\{0, \pm 1, \pm 2, \dots\}$	
rational numbers	$\mathbb{Q}$	$\{\frac{p}{q} \mid p, q \text{ are integers, } q \neq 0\}$	e.g. $\frac{1}{2}, \frac{3}{4}, 2 \in \mathbb{Q}$
irrational numbers	$\mathbb{I}$		$2 - \frac{2}{1} = \frac{2}{1} - \frac{2}{1} = \frac{0}{1} = 0$ $\rightarrow$ is in lowest terms: $\gcd(m, n) = 1$

Prove that $\sqrt{2}$ is irrational:

Proof: Assume that $\sqrt{2} = \frac{m}{n}$ ($m, n \neq 0$) in its lowest term $\gcd(m, n) = 1$

$2n^2 = m^2 \Rightarrow m^2$ is an even number $\Rightarrow m$ is even number

denote m by $2k$ ($k \in \mathbb{Z}$) $\Rightarrow 2n^2 = m^2 \xrightarrow{m=2k} 2n^2 = 4k^2 \Rightarrow n^2 = 2k^2 \Rightarrow n^2$ is even $\Rightarrow n$ is even.

$\Rightarrow m$ and n share a factor 2. A contradiction! $\Rightarrow \sqrt{2}$ is irrational

Decimal expansions: a number is rational if and only if its decimal expansion is a repeating (or terminating) block of digits

$$\text{e.g. } \frac{1}{2} = 0.5000\dots = 0.50$$

$$\frac{1}{3} = 0.333\dots = 0.\bar{3}$$

Properties of Real numbers

The set of $\mathbb{R}$ is a complete ordered field

Field properties:

1. Associativity of $+$ / $\cdot$
2. Distributivity
3. Identity element
4. Inverse element $\dots$ 见预备学期笔记

Completeness Property:

A field is complete if any non-empty set bounded from above has a least upper bound (supremum) (确界存在性). This means the Real line does not have gaps or holes.

Intervals

An "interval" is a subset of the real line that contains at least two numbers and contains all the real numbers lying between any two of its elements (中间不断)

- Open interval: Contains neither end point
- Half-open interval: Contains only one endpoint.

$$(a, b) = \{x \mid a < x < b\}$$

$$[a, b) = \{x \mid a \leq x < b\}, (a, b] = \{x \mid a < x \leq b\}$$

- Closed interval: Contains both endpoints.

$$[a, b] = \{x \mid a \leq x \leq b\}$$

Ex. $(3,5)$ is an interval.

$\mathbb{R}$ is an interval both open and closed interval

$(1,3) \cup [7,47]$. $\times$ 不连续

Q. 为什么实数集既是开区间也是闭区间?

A. 核心在于 $\mathbb{R}$ 没有边界点。区间的定义依赖于边界点的存在。开/闭区间的定义式: 若有边界点 $\Rightarrow \mathbb{R} = \dots$

(命题逻辑: 对于命题 $p \Rightarrow q$, 若 p 为 F, 无论 q 为 T 或 F, 命题均为 T)

Absolute value

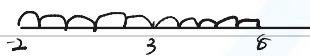
$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Geometry: $|x|$ = distance between x and 0.

$$|a+b| \leq |a|+|b|$$

Solve: $|x-3|=5$

Solution: find x s.t. $\overset{\text{distance}}{\uparrow} \text{dist}(x, 3) = 5$



Solve: $|2x-1| \leq 3$.

$$\frac{-5}{-3} \quad 0 \quad \frac{5}{3}$$

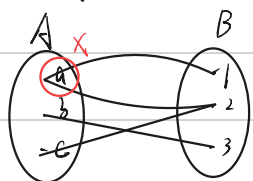
$$-3 \leq 2x-1 \leq 3$$

$$-2 \leq 2x \leq 4 \Rightarrow -1 \leq x \leq 2$$

Functions

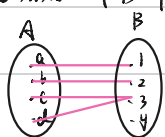
Def.

A function from set A to a set B ($f: A \rightarrow B$) is a rule that assigns to each element in A **exactly one element** $f(x)$ of B .



is not a function!

- The set A is called the domain ($\text{Dom} f$)
- If $x \in A$, $f(x)$ is the value of f at x
- In the notation $y = f(x)$, y is the dependent variable and x is the independent variable $y = f(x)$: f takes the value y at x .
- The subset of B of all possible values of f is called the range of f , $\text{ran} f$



$\text{Dom} = A$, $\text{ran} f = \{1, 2, 3\}$

ex. $A = \{1900, 1901, \dots, 2005\}$ $B = \mathbb{N}$ $f = \text{population of ST in the year } x$

ex. $A = \{C_r(p) : \text{circles on the plane}\}$ $C_r(p) = \bigcirc$

$B = \mathbb{R} > 0$

$f(C_r(p)) = (\text{the circumference of } C_r(p)) = 2\pi r$

$\text{ran} f = \mathbb{R} > 0$

ex. Let $A = \mathbb{R}$ $B = \mathbb{R}$

$f: A \rightarrow B$ $f(x) = x^2 + 1$ $f(2) = 2^2 + 1 = 5$ $f(-3) = (-3)^2 + 1 = 10$ $\text{Dom} f = \mathbb{R}$ $\text{ran} f = [1, +\infty) = \{x \geq 1\}$

Floor and ceiling function

Floor function $\lfloor x \rfloor = \{\text{the greater integer that is less or equal to } x\}$ ex. $\lfloor 3.2 \rfloor = 3$ $\lfloor -1.2 \rfloor = -2$

Ceiling function $\lceil x \rceil = \{\text{the smallest integer that is greater or equal to } x\}$ ex. $\lceil 3.1 \rceil = 4$ $\lceil -3.1 \rceil = -3$

Domain Convention

If a function is given by a formula and the domain is not stated explicitly, the convention (called 'the natural domain') is that the domain is the set **all the numbers for which the formula makes sense** (i.e. produces a real number)

General rules:

$\frac{f(x)}{g(x)}$: the denominator cannot be zero, so $g(x) \neq 0$

Square (even) root $\sqrt[k]{f(x)}$ The argument inside the root must be non-negative, so $f(x) \geq 0$

Eg. $f(x) = \frac{\sqrt{x-1}}{x^2-16}$

$$\begin{cases} \sqrt{x-1} \geq 0 \\ x^2-16 \neq 0 \end{cases} \Rightarrow \begin{cases} x \geq 1 \\ x \neq \pm 4 \end{cases} \Rightarrow \text{Dom}(f) = [1, 2) \cup (2, +\infty)$$

Eg. A force that depends on the distance ^① (距离非负)

注意规范背景

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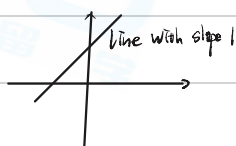
$F(x) = \frac{1}{x^2} \quad (x > 0) \Rightarrow \text{Dom}(f) = (0, +\infty)$

f: A → B A ⊆ ℝ, B ⊆ ℝ.

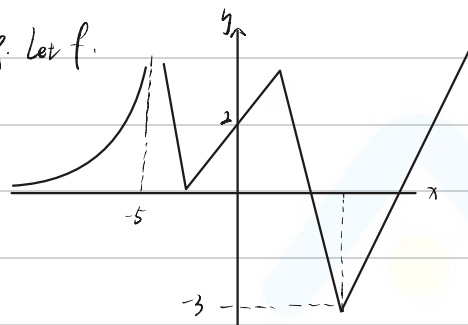
graph $\{(x, y) \in \mathbb{R}^2 : y = f(x)\} = \{(x, f(x)) \in \mathbb{R}^2\}$ ^{curve}

eg. $f(x) = x+1$

graph $(f) = \{(x, x+1)\} = \{(x, \frac{y}{x+1})\}$



eg. let f.



$\text{Dom}(f) = \mathbb{R} / (-5, 4)$

$\text{Ran}(f) = [-3, 2]$

$f(0) = 2$

Graph of functions:

The graph of f is a set of ordered pairs:

graph $(f) = \{(x, y) \in \mathbb{R}^2 : x \in \text{Dom}(f) \text{ and } y = f(x)\}$

Vertical line test: A curve _{曲线} in the plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Def.

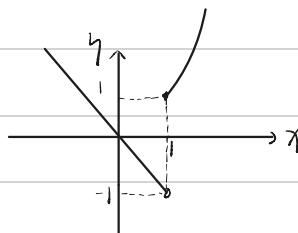
Piecewise defined function: 分段函数

It is defined by different formula in different part of each domain

E.g.

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ -x & \text{if } x < 1 \end{cases} \quad \begin{matrix} f(3) = 3^2 = 9 \\ f(-1) = -(-1) = 1 \end{matrix}$$

graph:



E.g. Absolute value function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Symmetry (even and odd function)

Even function: f is an even function if:

$$\forall x \in D, f(x) = f(-x)$$

e.g. $f(x) = x^2$, $\text{Dom}(f) = \mathbb{R}$

If $f(x) = x^{2n}$ ($n \in \mathbb{Z}$) $\rightarrow f(x)$ is even.

If f is an even function $\rightarrow$ the graph of f is symmetric with respect to y-axis. $(x, f(x)), (-x, f(x))$

Odd function: f is an odd function if:

$$\forall x \in D, f(-x) = -f(x)$$

e.g. $f(x) = x^3$

If $f(x) = x^n$ with n odd $\rightarrow f$ is an odd function

If f is an odd function, its graph is symmetric with respect to the origin $(x, f(x)), (-x, f(-x)) = (-x, -f(x)) = -(x, f(x))$

E.g. check if

$f(x) = x^3(x-1)(x+1)$ is even, odd or neither.

$$f(x) = x^3(x^2-1)$$

$$f(-x) = (-x)^3((-x)^2-1) = -x^3(x^2-1) = -f(x) \Rightarrow \text{odd}$$

Remark: f is odd and even at the same time

$$f(x) = f(-x) = -f(x) \Rightarrow 2f(x) = 0 \Rightarrow f(x) = 0 \text{ for all } x \in \mathbb{R}$$

Def.

We say that a function is an increasing function in an interval I if:

$$f(x_1) < f(x_2) \text{ for } x_1 < x_2, x_1, x_2 \in I$$

We say that a function is a decreasing function on I if:

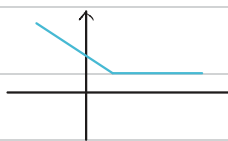
$$f(x_1) > f(x_2) \text{ for } x_1 < x_2, x_1, x_2 \in I$$

A function f is non decreasing if:

$$f(x_1) \leq f(x_2) \text{ for } x_1 < x_2, x_1, x_2 \in I$$

A function f is non increasing if:

$$f(x_1) \geq f(x_2) \text{ for } x_1 < x_2, x_1, x_2 \in I$$



non-increasing function
but not a decreasing function

Graph Transformations

Shifting a graph

- $y = f(x) + c$ shifts the graph of up if $c > 0$ or down if $c < 0$
- $y = f(x + c)$ shifts the graph of c units to the left if $c > 0$
- $y = f(x - c)$ shifts the graph of c units to the right if $c > 0$

Scaling (let $c > 1$)

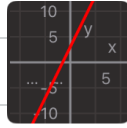
- $y = c \cdot f(x)$: Stretches vertically by a factor of c
- $y = \frac{1}{c} \cdot f(x)$: Compress vertically
- $y = f(c \cdot x)$: Compress horizontally
- $y = f(\frac{1}{c} \cdot x)$: Stretches horizontally

Catalogs of Functions

Linear functions: $y = kx + b \rightarrow$ ^{slope} y -intercepts

eg. $y = 2x + 3$ slope = 2, y -intercepts = 3

slope = $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$, $f(0) = k \cdot 0 + b = b$



Polynomials: A function $P(x)$ is called a polynomial if it's of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

Where n is a non-negative integer, and $a_0, \dots, a_n$ are real numbers called the coefficients of the polynomial.

If $a_n \neq 0$, the polynomial is of degree n .

A polynomial of degree 1 is a linear function.

A polynomial of degree 2 is called a quadratic function.

A polynomial of degree 3 is called a cubic function.

Rational functions: $f(x) = \frac{P(x)}{Q(x)}$, P, Q are polynomials

Where the domain is: $\text{Dom}(f) = \mathbb{R} - \{x : Q(x) = 0\}$

Power functions: $f(x) = x^a$ a is constant

1) $f(x) = x^n$ (n is a positive integer)

• If n is even $\rightarrow f$ is an even function

• If n is odd $\rightarrow f$ is an odd function

2) $f(x) = x^{\frac{1}{n}}$ (Root functions)

• If n is even: $\text{Dom}(x^{\frac{1}{n}}) = [0, +\infty)$, $\text{Ran}(x^{\frac{1}{n}}) = [0, +\infty)$ eg. $f(x) = x^{\frac{1}{2}} = \sqrt{x}$

• If n is odd: $\text{Dom}(x^{\frac{1}{n}}) = \mathbb{R}$, $\text{Ran}(x^{\frac{1}{n}}) = \mathbb{R}$ eg. $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$

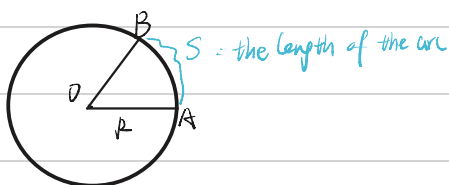
Reciprocal function: $f(x) = \frac{1}{x}$ $\text{Dom}(f) = \mathbb{R} - \{0\}$, $\text{Ran}(f) = \mathbb{R} - \{0\}$

Algebraic Functions: Algebraic functions are those constructed using algebraic operation ($+$, $-$, $\times$, $\div$, $\sqrt{}$, etc.).

Started from polynomials.

eg. $f(x) = \frac{\sqrt[3]{2x-5}}{x+5}$, $\text{Dom}(f) = (-\infty, +\infty)$

Measurement of angles in radians:

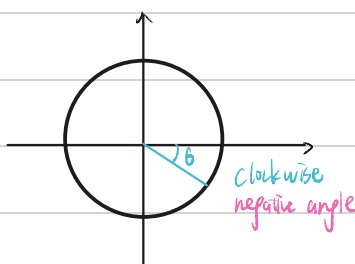
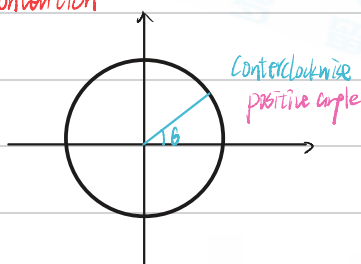


$\angle AOB$ measured in radians of this angle is: $\frac{s}{r} = \theta$ radians (θ)

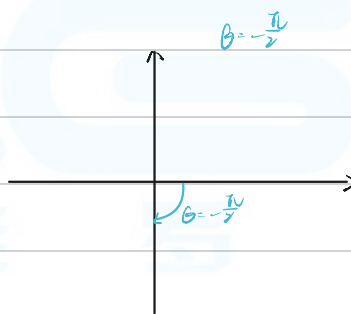
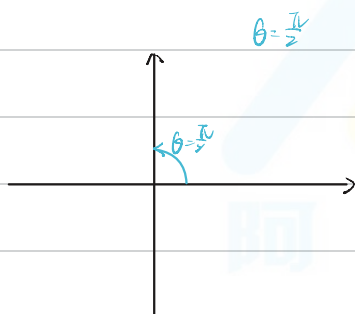
If $r=1$

$$s = r\theta \xrightarrow{\text{if } r=1} 2\pi r = r\theta \longrightarrow 2\pi = \theta = 360^\circ \longrightarrow \pi = 180^\circ$$

Convention:



eg.



Trigonometric function:

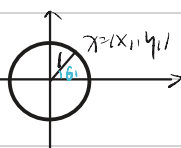
Cosine $\hookrightarrow \cos(x)$ (even f)

Sine $\hookrightarrow \sin(x)$ (odd f)

Range of $\cos$ and $\sin$ is $[-1, 1]$.

Periodicity: Both $\sin(x)$ and $\cos(x)$ are 2π -periodic:

$$\cos(x+2\pi) = \cos(x) \quad \sin(x+2\pi) = \sin(x)$$



$$x_1 = \cos \theta$$

$$y_1 = \sin \theta$$

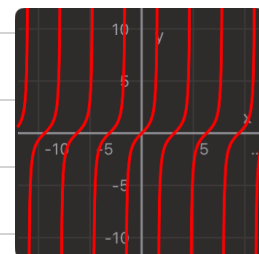
$$\cos\left(\frac{\pi}{2}\right) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

$$\text{tangent} \hookrightarrow \tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$D_{\tan} = \{x \mid \cos(x) \neq 0\} = \{x \mid x \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}\} \quad R_{\tan} = \mathbb{R} \quad (\text{odd f})$$

$$f(x) = \tan(x)$$



Trigonometric Identities

Pythagorean Theorem: $\cos^2 x + \sin^2 x = 1$, for all $x \in \mathbb{R}$

Cosine Addition: $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

Sine Addition: $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

Exponential Functions

Let $a > 0$, then

$$f(x) = a^x$$

is an exponential function

If $a > 1$, f is an increasing function.

If $0 < a < 1$, f is a decreasing function.

$$n \in \mathbb{N}, \quad a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n\text{-times}}, \quad a^{-n} = \underbrace{\frac{1}{a} \cdot \frac{1}{a} \cdots \frac{1}{a}}_{n\text{-times}} = \frac{1}{a^n}, \quad a^{\frac{1}{m}} = \sqrt[m]{a}, \quad a^{\frac{n}{m}} = \sqrt[m]{a^n} \quad (m, n \in \mathbb{Z})$$

Approach x by rational number $\frac{n}{m}$ (无限接近)

$$a^{\frac{n}{m}} \rightarrow a^x$$

Logarithm in base a .

For $a > 0$, $a \neq 1$, and $x > 0$. 指数函数非负

$$\log_a x = y \iff a^y = x$$

$$\text{Dom}(\log_a x) = \mathbb{R}^+ = \{x > 0\}$$

$$\text{Ran}(\log_a x) = \mathbb{R}$$

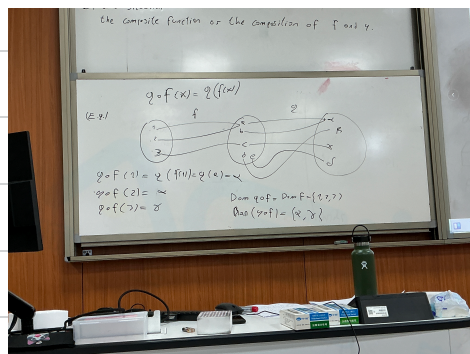
Composition of functions

Let f and g be two functions. If the range of f is a subset of the domain of g ($\text{Ran}(f) \subseteq \text{Dom}(g)$), the composition of g and f is the function defined by:

$$(g \circ f)(x) = g(f(x)) \quad \text{把 } f(x) \text{ 作为 } g \text{ 的自变量.}$$

满足 g 的定义域

The domain of $g \circ f$ is the set of all $x \in \text{Dom}(f)$ s.t. $f(x) \in \text{Dom}(g)$



eg. Let $f(x) = x+1$, $g(x) = \sqrt{x}$. $\text{Dom}(g \circ f) = ?$ $g \circ f(x) = \sqrt{x+1}$ $x \geq -1$

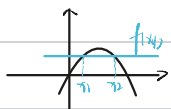
$$\text{Dom}(g \circ f) = \{x \mid x \in \text{Dom}(f) \text{ s.t. } f(x) \text{ belongs to } \text{Dom}(g)\}$$

One-to-one function: (Injective function): 单射函数

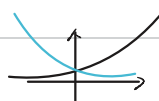
A function is called one-to-one function if it never takes on the same value twice.

That is, it satisfies $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ / $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Horizontal line test



$f(x_1) = f(x_2)$ not one-to-one (not injective)



If $f(x)$ is an increasing (decreasing) function, f is one-to-one

eg. Is the function: $f(x) = x^5 + 2$ one-to-one?

Sol. Let $x_1, x_2 \in \text{Dom}(f)$

$$\text{Assume } f(x_1) = f(x_2) \Rightarrow x_1^5 + 2 = x_2^5 + 2 \Rightarrow (x_1)^5 = (x_2)^5 \Rightarrow \sqrt[5]{x_1^5} = \sqrt[5]{x_2^5} \Rightarrow x_1 = x_2$$

f is an one-to-one / injective function

Conclusion: If f is one-to-one, for any $y \in \text{Ran}(f)$, there exists a unique $x \in \text{Dom}(f)$ st. $f(x) = y$.

Inverse Function (f^{-1})

（7课笔记有补充）

If f is a one-to-one function with domain A and range B , the Inverse function f^{-1} has domain B and range A defined by:

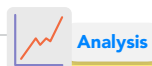
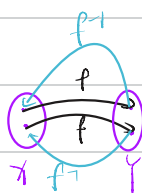
$$f^{-1}(y) = x \text{ if } f(x) = y.$$

The inverse function satisfies:

$$f(f^{-1}(y)) = y \text{ for all } y \in \text{Ran}(f)$$

$$f^{-1}(f(x)) = x \text{ for all } x \in \text{Dom}(f)$$

反函数的定义域是原函数的值域



Analysis

Find an Inverse function:

1) if $f(x)$ is one-to-one?

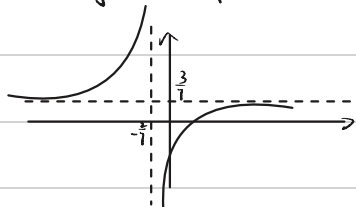
2) $y = f(x) \Rightarrow$ 整理为 $x = \dots y$ 形式

3) $f^{-1}(y) = \dots x$

$y = \dots x$ 说明了 x 是怎么运算得到 y 的。; $x = \dots y$ 说明了 y 可以如何运算从而得到 x 。此时把 y 中 x 的变量, 替换为自变量, 若原函数是单射的, 经过其相反路径的运算, 一定可以回到原函数的 x 。

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eg. Let $f(x) = \frac{3x-5}{7x+2}$



$$\text{Dom}(f) = \mathbb{R} - \{-\frac{2}{7}\}$$

$$7xy - 3x = -5 - 2y$$

$$(7x+2)y = 3x-5$$

$$(7y-3)x = -5-2y$$

$$7xy+2y=3x-5$$

If $y \neq \frac{3}{7} \Rightarrow x = \frac{-5-2y}{7y-3} \Rightarrow \exists! x \text{ s.t. } f(x)=y$ where $x = \frac{-5-2y}{7y-3} = f^{-1}(y)$ by definition

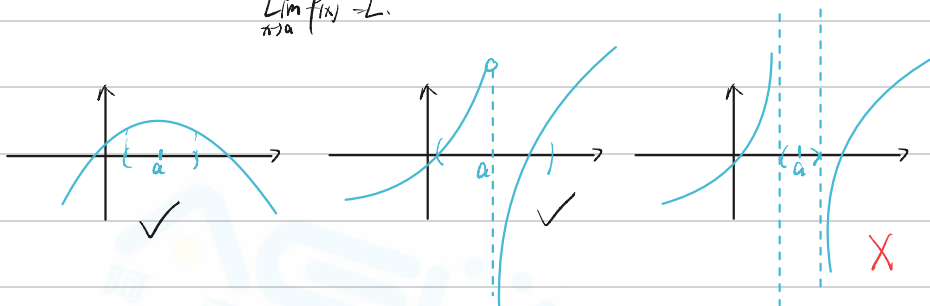
exists an unique

Limit of function

Intuitive Definition

Let f be a function defined on some open interval, except possibly at a . We say that the limit of a function f as x approaches a equals L and we write:

$$\lim_{x \rightarrow a} f(x) = L$$



开区间中要包含 a . 在这段区间中 只有 a 这个点, 可以不在定义域内

If we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a (on either side) but not equal to a . Sometimes we will write $f(x) \rightarrow L$ as $x \rightarrow a$. (强调逐渐靠近)

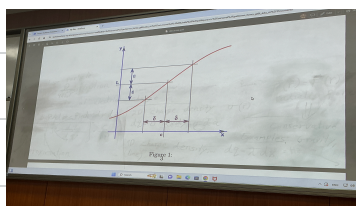
Formal Definition

The limit of function f as x approaches a is L , written as

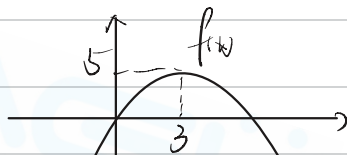
$$\lim_{x \rightarrow a} f(x) = L$$

If, for every number $\epsilon > 0$, (eps lon, which measures closeness on the y -axis), there exists a number $\delta > 0$ (delta, which measures closeness on x -axis) s.t.

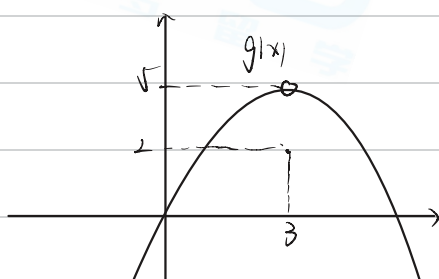
If $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$ (x 与 a 无限接近, y 与 L 无限接近)



e.g.



$$\lim_{x \rightarrow 3} f(x) = 5$$



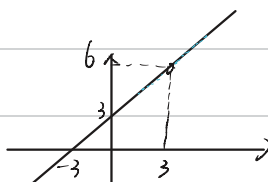
$$\lim_{x \rightarrow 3} g(x) = 5$$

函数极限并不一定是此时函数值, 要看函数在该点是否连续.

eg. Consider $f(x) = \frac{x^2 - 9}{x - 3}$, $\lim_{x \rightarrow 3} f(x) = 6$

If $x \neq 3$.

$$f(x) = \frac{(x+3)(x-3)}{(x-3)} = x+3.$$



eg. Let consider

$$f(x) = \sin\left(\frac{\pi}{x}\right) \quad \text{Dom}(f) = \mathbb{R} - \{0\}$$

$$\lim_{x \rightarrow 0} f(x) = ?$$

Sol

$$x_n = \frac{1}{n} \quad (n \in \mathbb{N}) \quad \left\{1, \frac{1}{2}, \frac{1}{3}, \dots \rightarrow 0\right\} \quad \text{(approaches 0 from right)}$$

$$\tilde{x}_n = \frac{2}{2n+1} \quad (n \in \mathbb{N}) \quad \left\{\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \dots \rightarrow 0\right\} \quad \text{(approaches 0 from right)}$$

Compute

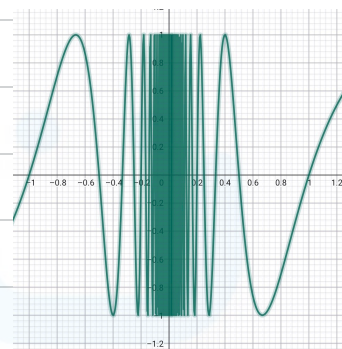
$$f(x_n) = \sin\left(\frac{\pi}{\frac{1}{n}}\right) = \sin(n\pi) = 0 \quad \text{for } n \in \mathbb{N}$$

$$f\left(\frac{2}{2n+1}\right) = \sin\left(\frac{\pi}{\frac{2}{2n+1}}\right) = \sin\left[\left(\frac{2n+1}{2}\right)\pi\right] = 1 \text{ or } -1 \quad \text{for } n \text{ is even, } -1 \text{ for } n \text{ is odd.}$$

When

$$x_n \rightarrow 0 \quad f(x_n) = 0$$

$$\tilde{x}_n \rightarrow 0 \quad f(\tilde{x}_n) = \pm 1$$



当 x 以不同方式同方向趋近 0 时, $f(x)$ 趋向不同值, 故 $f(x)$ 极限不存在

$$f(x) = \sin\left(\frac{\pi}{x}\right) \quad x \rightarrow 0 \quad \text{剧烈振荡}$$

Limit Laws

★前提: 涉及的函数存在极限!!!

A $f(x) = c = \text{const.} \quad \lim_{x \rightarrow a} c = c$

B $f(x) = x \quad \lim_{x \rightarrow a} x = a$

Assume $f(x)$ and $g(x)$: $\lim_{x \rightarrow a} f(x) = L_1$ $\lim_{x \rightarrow a} g(x) = L_2$ (exists) $\rightarrow$ number.

① $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L_1 + L_2$

② $\lim_{x \rightarrow a} (c \cdot f(x)) = c \cdot \lim_{x \rightarrow a} f(x) = cL_1$ (③ + ④)

③ $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \left[\lim_{x \rightarrow a} f(x)\right] \cdot \left[\lim_{x \rightarrow a} g(x)\right] = L_1 \cdot L_2$

④ $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L_1}{L_2}$ provided $\lim_{x \rightarrow a} g(x) = L_2 \neq 0$

⑤ $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x)\right]^n = L_1^n$

⑥ $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L_1}$ If n is even, we must have $\lim_{x \rightarrow a} f(x) \geq 0$

C. Let $p(x)$ be polynomial

$$\lim_{x \rightarrow a} p(x) = p(a)$$

eg. $p(x) = 3x^2 + x + 1$

$$\lim_{x \rightarrow 2} (3x^2 + x + 1) = ?$$

$$\lim_{x \rightarrow 2} (3x^2) = 3 \lim_{x \rightarrow 2} x^2 = 3 \lim_{x \rightarrow 2} x \cdot \lim_{x \rightarrow 2} x = 3 \cdot 2 \cdot 2 = 12$$

$$\lim_{x \rightarrow 2} (3x^2) = c \lim_{x \rightarrow 2} f(x) \quad \lim_{x \rightarrow 2} x = a$$

$$\lim_{x \rightarrow 2} x = 2$$

$$\lim_{x \rightarrow 2} 1 = 1$$

$$\lim_{x \rightarrow 2} c = c$$

$$\lim_{x \rightarrow 2} (3x^2 + x + 1) = \lim_{x \rightarrow 2} (3x^2) + \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 1$$

Eg. Forcb

$$\lim_{x \rightarrow 2} \sqrt[3]{x^3 + 7x + 6} = \sqrt[3]{\lim_{x \rightarrow 2} (x^3 + 7x + 6)} = \sqrt[3]{24}$$

$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$
 $\lim_{x \rightarrow a} p(x) = p(a)$

Eg. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ $\lim_{x \rightarrow 1} x \neq 0 \Rightarrow$ we cannot use the quotient law ($\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, where $\lim_{x \rightarrow a} g(x) \neq 0$)

If $x \neq 1, \frac{x^2 - 1}{x - 1} = x + 1$

D. If $f(x) = g(x)$, when $x \neq a$.

$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ (常可同义或限制, eg. $f(x) = \frac{x^2 - 1}{x - 1}, g(x) = x + 1$)

Eg. (for D)

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x}$$

If $x \neq 0$.

$$f(x) = \frac{\sqrt{x^2 + 4} - 2}{x} = \frac{\sqrt{x^2 + 4} - 2}{x} \cdot \frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2} = \frac{(\sqrt{x^2 + 4})^2 - 2^2}{x(\sqrt{x^2 + 4} + 2)} = \frac{x^2}{x(\sqrt{x^2 + 4} + 2)} = \frac{x}{\sqrt{x^2 + 4} + 2} = g(x)$$

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$

Comparison Law

Let $f(x)$ and $g(x)$ be two functions defined on an open interval that contains a , except possibly at a . Assume that $f(x) \leq g(x)$ for any x near a , (except possibly at a), and that the limits of f and g exists as $x \rightarrow a$. Then:

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

Expansion

Sandwich Theorem

Assume that in an interval I that contains a s.t.

$$g(x) \leq f(x) \leq h(x), \text{ for all } x \text{ in } I, \text{ maybe not at } a.$$

补充: 三明治定理的单独形式 ($x \rightarrow a^+, x \rightarrow a^-$)

如果对于 x 在 C 的右侧 ($C < x < C + \delta$) 有

Also assume that

(去掉个别点的整体趋势)

$$g(x) \leq f(x) \leq h(x)$$

$$\lim_{x \rightarrow a} g(x) = L = \lim_{x \rightarrow a} h(x)$$

(f 夹在 g, h 中间且 g, h 收敛或发散)

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$$

若 $\lim_{x \rightarrow a} g(x) = L_1$

Then:

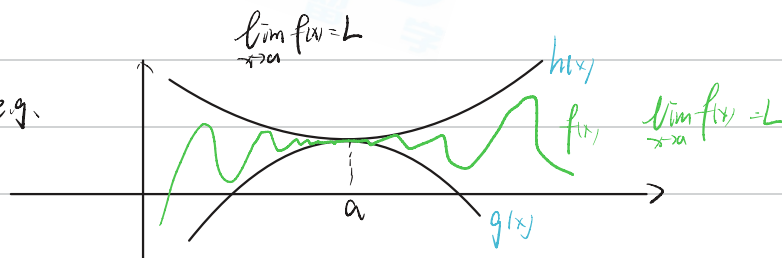
不收敛的整体的趋势

$$\lim_{x \rightarrow a} f(x) = L$$

$\lim_{x \rightarrow a} h(x) = L_2, L_1 \neq L_2$

在区间内, 若 $C - \delta < x < C$ 有

e.g.



$$g(x) \leq f(x) \leq h(x)$$

若 $\lim_{x \rightarrow a} f(x)$ 存在, 则

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L$$

$\lim_{x \rightarrow a} f(x)$ 处于 L_1, L_2 之间

$$\lim_{x \rightarrow a} f(x) = L$$

E.g. $\lim_{x \rightarrow 0} x^2 \sin(\frac{1}{x})$

If $x \neq 0$

$$-1 \leq \sin(\frac{1}{x}) \leq 1$$

$$-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$$

$$\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} (-x^2) = 0 \xrightarrow{\text{Sandwich theorem}} \lim_{x \rightarrow 0} x^2 \sin(\frac{1}{x}) = 0$$

一个不存在极限但有界的函数 $\times$ 极限存在的函数，常用三明治定理，且所有极限通常存在。

Precise Definition of a Limit.

Precise Definition of a Finite Limit

The limit of a function $f(x)$ as x approaches a is L , written as $\lim_{x \rightarrow a} f(x) = L$.

If for every number $\epsilon > 0$, there exists a number $\delta > 0$, s.t.

$$\text{If } 0 < x - a < \delta, \text{ then } |f(x) - L| < \epsilon$$

e.g. Prove $\lim_{x \rightarrow 6} \frac{x}{2} + 1 = 4$.

take $\epsilon = \frac{1}{2}$ $|f(x) - 4| < \epsilon$

iff $|\frac{x}{2} + 1 - 4| < \frac{1}{2} \Rightarrow |\frac{x}{2} - 3| < \frac{1}{2} \Rightarrow |x - 6| < 1 \rightarrow \delta$

$$|f(x) - L| < \epsilon$$

$$|\frac{x}{2} + 1 - 4| < \epsilon$$

$$|\frac{x}{2} - 3| < \epsilon$$

$$|x - 6| < 2\epsilon \rightarrow \delta$$

We choose that $\delta = 2\epsilon$, s.t.

$$0 < |x - 6| < \delta = 2\epsilon$$

$$|f(x) - 4| = |\frac{x}{2} - 3| = \frac{1}{2}|x - 6| < \frac{1}{2} \cdot 2\epsilon = \epsilon$$

逆推顺证

E.g. Prove: $\lim_{x \rightarrow 2} x^2 - 6x + 9 = 1$.

$$|f(x) - 1| < \epsilon$$

$$|x^2 - 6x + 9| = |x^2 - 6x + 8| = |(x-2)(x-4)| < \epsilon$$

Assume that $\delta \leq 1$ (找到对应的 δ 证明, 此限制用来方便计算)

$0 < |x - 2| < \delta$ 是 $x \rightarrow 2$ 时的极限

$$0 < |x - 2| < 1$$

$$\downarrow$$

$$|x - 4| < 3$$

$$\text{If } |x - 2| < 1 \rightarrow |f(x) - 1| = |(x-2)(x-4)| = |x-2||x-4| < 3|x-2|$$

Given $\epsilon > 0$ $\delta = \text{minimum}(1, \frac{\epsilon}{3})$ (choose smaller one)

Let x , s.t.

$$0 < |x - 2| < \delta$$

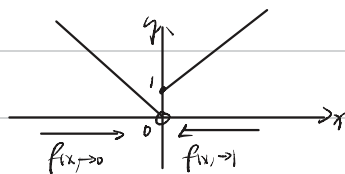
$$|f(x) - 1| = |x-2||x-4| < 3|x-2| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

this prove $\lim_{x \rightarrow 2} x^2 - 6x + 9 = 1$

Side Limits — Continuity

E.g.

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) \Rightarrow \lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$

One-Sided Limits

Definition: We say that the "left-hand limit" of $f(x)$ as x approaches a is L , and we write

$$\lim_{x \rightarrow a^-} f(x) = L$$

If we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x < a$. (i.e. from the left)

Definition: We say that the "right-hand limit" of $f(x)$ as x approaches a is L , and we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

If we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x > a$. (i.e. from the right)

In the previous example, by definition, $\lim_{x \rightarrow 0^-} f(x) = 0$, $\lim_{x \rightarrow 0^+} f(x) = 1$

E.g. $f(x) = \frac{x^2}{|x|}$ (left-limit) $\lim_{x \rightarrow 0^-} f(x) = ?$, (right-limit) $\lim_{x \rightarrow 0^+} f(x) = ?$

Sol. If $x < 0$.

$$f(x) = \frac{x^2}{|x|} = \frac{x^2}{-x} = -x$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -x = 0$$

If $x > 0$

$$f(x) = \frac{x^2}{|x|} = \frac{x^2}{x} = x$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

Relationship to Two-Sided Limit

Let f be a function defined in an interval that contains a (possibly not at a). Then we have

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

1) $\lim_{x \rightarrow a} f(x)$ exists $\Rightarrow \lim_{x \rightarrow a} f(x) = L$
 极限存在 $\Rightarrow$ 极限值

Precise Definition of Sided-Limits.

Right-Limit: Let f defined in an interval (a, b) . The "right-hand limit":

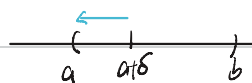
$$\lim_{x \rightarrow a^+} f(x) = L$$

If any $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$|f(x) - L| < \epsilon \text{ whenever } x \in (a, a + \delta)$$

$$\downarrow$$

$$0 < x - a < \delta$$



Left-Limit: Let f defined in an interval (c, a)

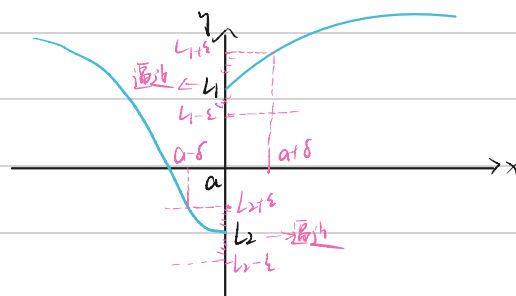
$$\lim_{x \rightarrow a^-} f(x) = L$$

If for any $\epsilon > 0$, $\exists \delta > 0$.

$$|f(x) - L| < \epsilon \text{ if } x \in (a - \delta, a)$$

$$\downarrow$$

$$0 < a - x < \delta$$



E.g. Given $f(x) = x^2 + 1$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

$$\text{Let } |f(x) - 1| = |x^2 + 1 - 1| < \epsilon$$

$$\text{iff } |x^2| < \epsilon$$

$$\text{iff } x^2 < \epsilon \Rightarrow -\sqrt{\epsilon} < x < \sqrt{\epsilon}$$

$$0 < x < \sqrt{\epsilon} \quad \text{we want non-negative number to prove right-hand limit}$$

We can take $\delta = \sqrt{\epsilon}$

$$|f(x) - 1| = |x^2 + 1 - 1| = |x^2| < (\sqrt{\epsilon})^2 = \epsilon$$

$$\text{Thus, } \lim_{x \rightarrow 0^+} f(x) = 1$$

A Special Trigonometric Limit

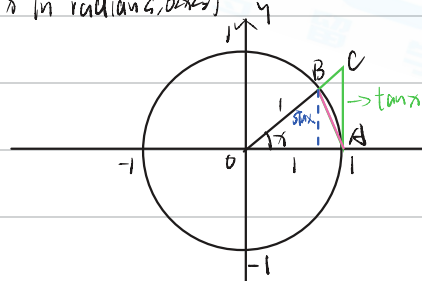
$$\star \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Proof: Strategy:

$$1) \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 \Rightarrow \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$2) \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1$$

(in radians, $0 < x < \frac{\pi}{2}$)



Section of the disk OAB

$$\text{Area of Disk} = \pi r^2$$

$$\text{Area of } \triangle OAB = \frac{1}{2} \cdot \pi r^2 = \frac{\pi}{2}$$

Triangle $\triangle OAB$ (small one)

and (big one)

$$S_{\triangle OAB} \leq S_{\text{section}} \leq S_{\triangle OAC} \Rightarrow \frac{1}{2} \cdot 1 \cdot \sin x < \frac{\pi}{2} < \frac{1}{2} \cdot 1 \cdot \tan x$$

(by graph)

Then, for any $0 < \epsilon < \frac{\pi}{2} \Rightarrow \sin x, \cos x > 0$

$$\frac{\sin x}{x} < \frac{x}{2} < \frac{\tan x}{2} \quad / \cdot x / \sin x$$

$$\downarrow$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$\text{giving } \boxed{\cos x} < \boxed{\frac{\sin x}{x}} < \boxed{1} \quad \text{h.w. } \star \text{ Sandwich theorem}$$

$$\text{Since } \lim_{x \rightarrow 0^+} \cos x = \lim_{x \rightarrow 0^+} 1 = 1$$

$$\text{Thus, } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} h(x) \cdot \lim_{x \rightarrow 0^+} g(x) = 1 \quad (\text{we have solved step 1})$$

Since $\sin(x)$, x are odd function

Thus, $\frac{\sin x}{x}$ is even function.

$$\text{Thus, } \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

E.g. $\lim_{x \rightarrow 0} \frac{\sin(4x)}{7x}$

$$= \lim_{x \rightarrow 0} \frac{\frac{4}{7} \sin(4x)}{4x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{4}{7} \sin(4x)}{4x} = \boxed{\frac{4}{7}} \lim_{x \rightarrow 0} \frac{\sin(4x)}{4x} \quad \text{Limit Law} \quad \frac{\sin x}{x} \rightarrow 1$$

Since $x \rightarrow 0 \Rightarrow 4x \rightarrow 0$

Let $4x = \theta$

$$\frac{4}{7} \lim_{x \rightarrow 0} \frac{\sin(4x)}{4x} = \frac{4}{7} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{4}{7} \times 1 = 1$$

E.g. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos^2 x}{1 + \cos x}}{x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} = 1 \times 0 \times \frac{1}{2} = 0$$

Limit Law, Multiplication Law, $\lim_{x \rightarrow 0} \sin x = 0$, $\lim_{x \rightarrow 0} \cos x = 1$

Continuity

A function is continuous if its graph has no breaks, jumps, or holes. Formally:

Definition: A function is "continuous" at $x=a$ if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Remark: This definition requires that the following three conditions hold

1. $f(a)$ is defined (i.e. a is in the domain of f)
2. $\lim_{x \rightarrow a} f(x)$ exists (i.e. the left-hand and right-hand limits are equal)
3. The limit value **must** equal the function value: $\lim_{x \rightarrow a} f(x) = f(a)$

E.g. $f(x) = |x|$ is continuous at $x=0$.

① $0 \in \text{Dom}(f)$

② $f(0) = |0| = 0 \quad \xrightarrow{0=0=0} f \text{ is continuous at } x=0$

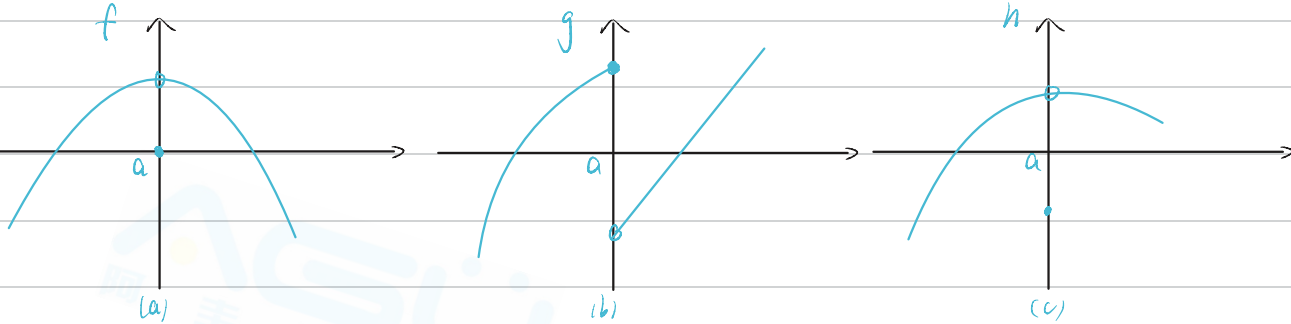
③ $\lim_{x \rightarrow 0} |x| = \lim_{x \rightarrow 0^+} x = 0, \quad \lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} -x = 0 \Rightarrow \lim_{x \rightarrow 0} |x| = 0$

E.g. Prove that $f(x) = \sqrt[3]{x^2 - 1}$ is continuous at $x=0$

Proof: 1) $\text{Dom}(f) = \mathbb{R}$, $0 \in \text{Dom}(f)$

$$2) \lim_{x \rightarrow 0} \sqrt[3]{x^2 - 1} = \sqrt[3]{\lim_{x \rightarrow 0} (x^2 - 1)} = \sqrt[3]{\lim_{x \rightarrow 0} x^2 - 1} = -1 \xrightarrow{0+0+0} f(x) \text{ is continuous at } x=0$$

$$3) f(0) = -1 = \lim_{x \rightarrow 0} f(x)$$



Some discontinuous function.

Discontinuity

Definition: A function f is "discontinuous at a " (or has a "discontinuity" at a) if it is not continuous at a .

Namely, this happens if **one or more** of the three conditions above fail.

Case 1: f is not defined at $x=a$.

e.g. f and h above

Case 2: $\lim_{x \rightarrow a} f(x) \neq f(a) \Rightarrow$ removable discontinuity (This discontinuity can be "removed" by redefining $f(a)$ to be equal to $\lim_{x \rightarrow a} f(x)$.)

Case 3: $\lim_{x \rightarrow a} f(x)$ does not exist $\Rightarrow$ Jump Discontinuity (The function "jumps" from one value to another at $x=a$. i.e. $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ e.g. g above.)

Case 4 Infinite Discontinuity: This occurs when one or both of the one-sided limits is ∞ or $-\infty$.

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow a^+} f(x) = \pm\infty$$

This type of discontinuity corresponds to a "Vertical Asymptote" at $x=a$.

Case 5 Oscillating Discontinuity: This occurs if the limit does not exist due to the function values oscillating

infinitely many times between two or more values as x approaches a

E.g. $f(x) = \sin \frac{1}{x}$ at $x=0$

Continuity From the Right and Left

Definition A function f is "continuous from the right at $x=a$ " if:

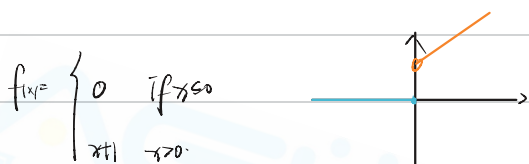
$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

A function f is "continuous from the left at $x=a$ " if:

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

Remark: A function is continuous at $a \iff$ it is continuous from both right and left at a .

E.g.



• $\lim_{x \rightarrow 0} f(x)$ does not exist, f is not continuous at $x=0$.

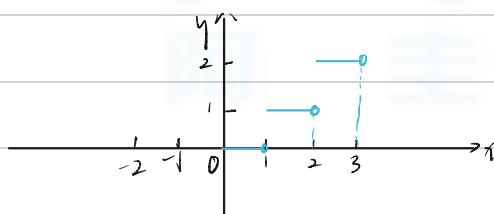
• $f(x)$ is left-continuous at $x=0$!

$$f(0)=0, \quad \lim_{x \rightarrow 0^-} f(x)=0$$

• $f(x)$ is not right continuous at $x=0$

$$\lim_{x \rightarrow 0^+} f(x) = 1 \neq f(0) = 0$$

E.g. Let $\lfloor x \rfloor$ be the floor function



$\lfloor x \rfloor$ is not continuous at any integer

Let z an integer

$$\lim_{x \rightarrow z^-} \lfloor x \rfloor = z-1$$

$$\lim_{x \rightarrow z^+} \lfloor x \rfloor = z \quad \Rightarrow \quad \lim_{x \rightarrow z} \lfloor x \rfloor \text{ does not exist at any integer}$$

• $\lfloor x \rfloor$ is not left-continuous at any integer!

• $\lfloor x \rfloor$ is right-continuous at any $x \in \mathbb{R}$!

E.g. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is not continuous at any point

Continuity on an Interval.

Definition A function f is continuous on an interval if it is continuous at any number in the interval.

f is continuous on $I=(a,b)$ if f is continuous at any $x \in (a,b)$

f is continuous on $I=[a,b]$ if 1) f is continuous on (a,b)

2) f is right-continuous at a .

f is left-continuous at b .

E.g. Prove that $f(x) = \sqrt{1-x^2}$ is continuous on $[-1, 1]$

- 1) $x \in (-1, 1)$ $\lim_{x \rightarrow a} f(x) = f(a) = \sqrt{1-a^2}$, f is continuous on $(-1, 1)$ ①
 - 2) $\lim_{x \rightarrow -1^+} f(x) = 0 = f(-1)$ f is right continuous at -1 ②
 - $\lim_{x \rightarrow 1^-} f(x) = 0 = f(1)$ f is left continuous at 1 ③
- f is continuous on $[-1, 1]$

Continuity Theorems.

Theorem: If f and g are continuous at a , and c is constant, then the following functions are also continuous at a :

1) $(cf)(x)$

Proof: Since f, g are continuous at a .

Thus, $\lim_{x \rightarrow a} f(x) = f(a)$, $\lim_{x \rightarrow a} g(x) = g(a)$ (by the definition of continuity).

$$\lim_{x \rightarrow a} (cf)(x) = \lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x) = c f(a) = (cf)(a)$$

2) $(f)(x)$

3) $(f \cdot g)(x)$

4) $(\frac{f}{g})(x)$, provided that $g(a) \neq 0$

5) $f^n(x)$ and $\sqrt[n]{f(x)}$

Corollaries:

1) $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ (a polynomial) is continuous 2+5

2) $f(x) = \frac{P(x)}{Q(x)}$ rational function, f is continuous in its domain (namely $Q(x) \neq 0$)

E.g. $f(x) = \frac{1}{x}$ is not continuous on $\mathbb{R}$. However, it's continuous on its domain: $\mathbb{R} \setminus \{0\}$

3) $\cos(x)$ and $\sin(x)$ are continuous at x_0 .

Proof: We have to prove $\lim_{x \rightarrow x_0} \cos(x) = \cos(x_0)$

$$x = z + x_0, \text{ where } z = x - x_0$$

$$\lim_{x \rightarrow x_0} \cos(x) = \lim_{z \rightarrow 0} \cos(z + x_0) = \lim_{z \rightarrow 0} [\cos(z) \cos(x_0) - \sin(z) \sin(x_0)] = \lim_{z \rightarrow 0} \cos(z) \cos(x_0) - \lim_{z \rightarrow 0} \sin(z) \sin(x_0) = \cos(x_0) \lim_{z \rightarrow 0} \cos(z) - \sin(x_0) \lim_{z \rightarrow 0} \sin(z) = \cos(x_0)$$

Similar to $\sin(x)$

Theorem: Let f be defined on an interval I . Also, assume that f is a one-to-one function (injective function) and continuous in I .

Then:

f^{-1} is also continuous (try to prove)

E.g. 1) $\tan(x) = \frac{\sin(x)}{\cos(x)}$ $(-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow \mathbb{R}$

$\tan^{-1}(x) = \arctan(x) : \mathbb{R} \Rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ is continuous

2) $f(x) = \ln x + \sqrt{x}$ is continuous.

Dom(f) = $\mathbb{R}^+$.

$y = \frac{\sqrt{x^2 - x}}{\sqrt{x}} \quad x > 0$

Theorem: If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then
 $\lim_{x \rightarrow a} f(g(x)) = f(b)$

$x \rightarrow a$

$\downarrow$

$g(x) \rightarrow b \quad (x' \rightarrow b) + f \text{ is conti.}$

$\downarrow$

$f(g(x)) = f(b)$

E.g. Find $\lim_{x \rightarrow 0^+} \ln \left(\frac{\sqrt{x^2 - x}}{\sqrt{x}} \right)$ ①

② When $x \rightarrow 0$.

$\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 - x}}{\sqrt{x}} = \lim_{x \rightarrow 0^+} 1 - \sqrt{x} = 1$

$f(x) = \frac{\sqrt{x^2 - x}}{\sqrt{x}} = 1 - \sqrt{x} = g(x)$ ②

$\ln$ is continuous function at $x=1$

$\ln$ is continuous function.

$\lim_{x \rightarrow 0^+} \ln \left(\frac{\sqrt{x^2 - x}}{\sqrt{x}} \right) = \ln \left(\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 - x}}{\sqrt{x}} \right) = \ln(1) = 0$

Theorem: If f is continuous at $g(a)$ and g is continuous at a , then.

$f \circ g$ is continuous at a . (连续性的传递)

E.g. Is $f(x) = \tan^{-1}(x^2 + x \cos^3(x))$ continuous?

$x^2 + x$; $\cos(x)$ is continuous

$\downarrow$
continuous (polynomial)

$\downarrow$
 $\cos^3(x)$ is continuous

$\Rightarrow x^2 + x \cos^3(x)$ is continuous.

And $\tan^{-1}$ is continuous. (1. $\tan$ is defined in $(-\frac{\pi}{2}, \frac{\pi}{2})$. 2. injective 3. continuous $\Rightarrow \tan^{-1}$ is continuous)

$\downarrow$
 $f(x)$ is continuous

Intermediate Value Theorem

Let f be a continuous function defined on an interval $[a, b]$. Let N be a number between $f(a)$ and $f(b)$.

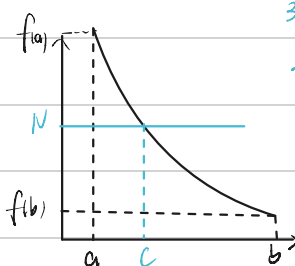
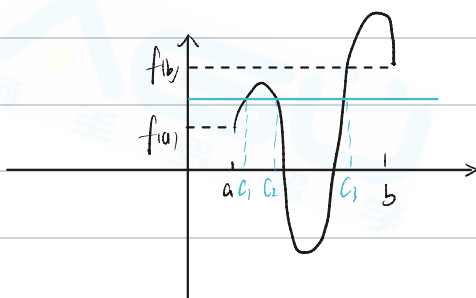
Then, there exists $c \in [a, b]$ s.t. $f(c) = N$

连续函数

$f(0)=3, f(1)=2$, continuous on an open interval $(1, 2)$. $f(0) > 2 > f(1)$

But there is no x_0 s.t. $f(x_0) = 2.1$

Counterexample if the interval is open.



f takes on all values in between $f(a)$ and $f(b)$