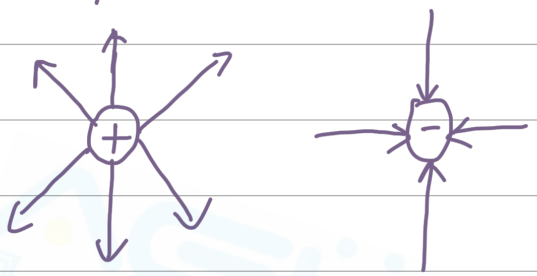
3.7

Coulomb 库仑's law $F = \frac{kq_1q_2}{r^2}$ $k = 9\times10^9 N\frac{m^2}{C^2}$ $k = \frac{1}{4\pi\varepsilon_0}$

$\vec{F}_{1\to2} = \frac{kq_1q_2(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3}$

$\vec{E}$



强度：density

$\vec{E} = \frac{\vec{F}}{q} = \frac{kq}{r^2}\hat{r}$ If the point charge is not at origin

$\vec{E} = \frac{\frac{kqq_0(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}}{q_0} = \frac{kq(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$ $\vec{E} = \sum_{i=1}^{N}\frac{kq_i(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$ Vector sum

3.8 物理丁

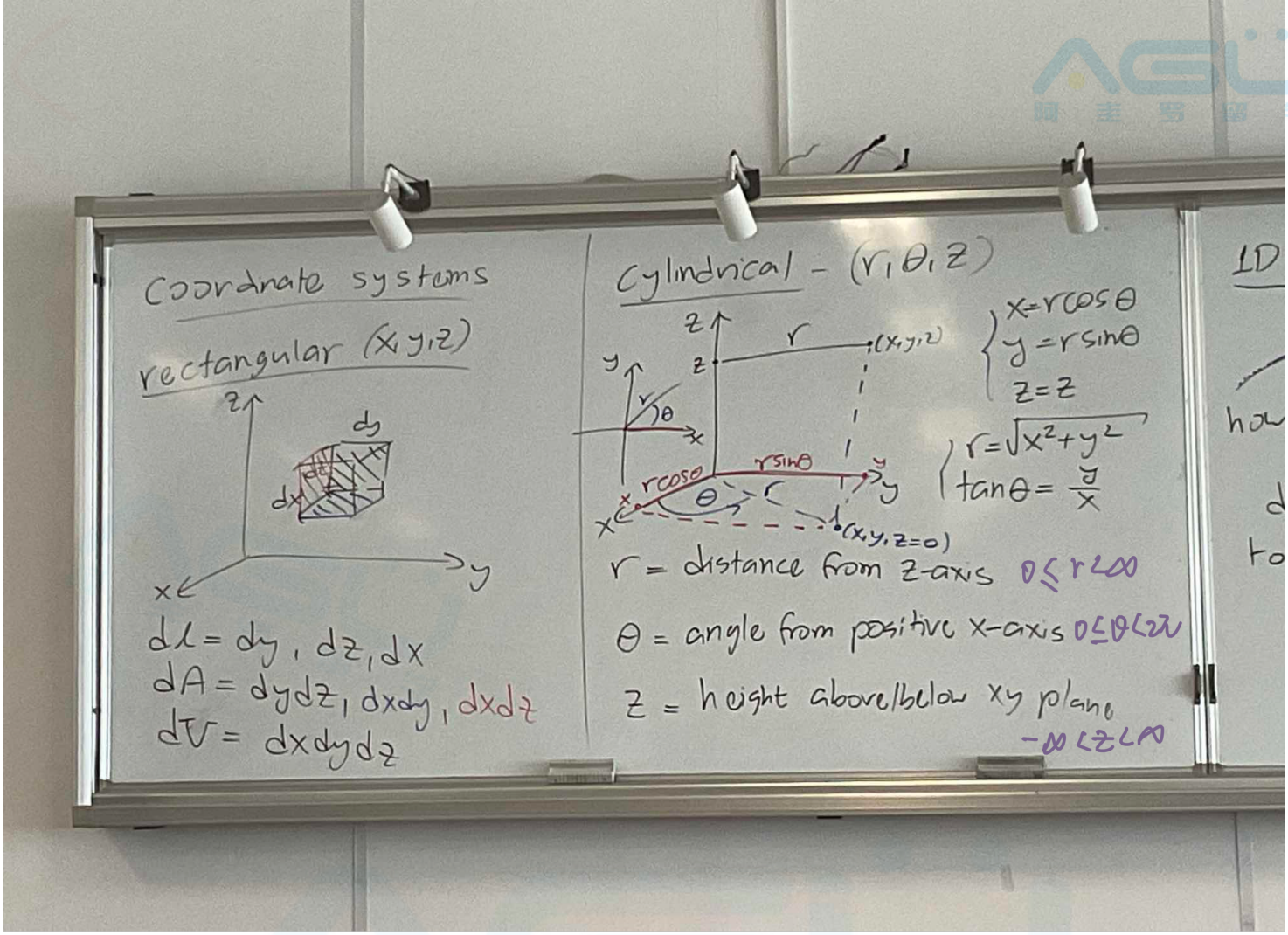
total charge

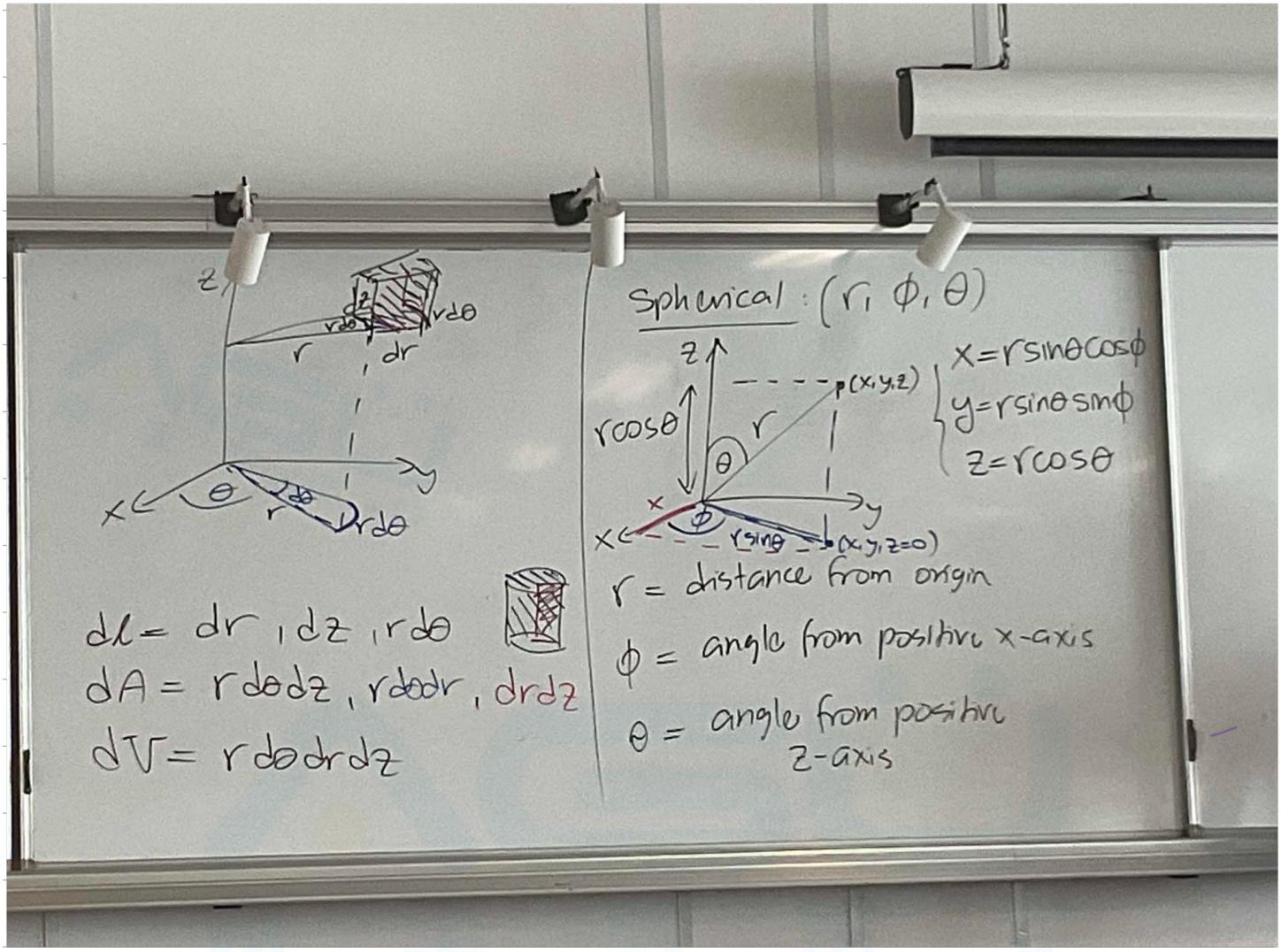
1D Line charge density λ (how much charge per unit length) $\lambda = \frac{dq}{dl}$

$dq = \lambda dl$ total charge $Q_{tot} = \int dq = \int \lambda dl$

2D $\sigma = \frac{dq}{dA}$ $Q_{tot} = \int dq = \iint \sigma dA = \iint \sigma dxdy$ double integral

3D $\rho = \frac{dq}{dv}$ $Q_{tot} = \int dq = \iiint \rho dv = \iiint \rho dxdydz$ triple integral

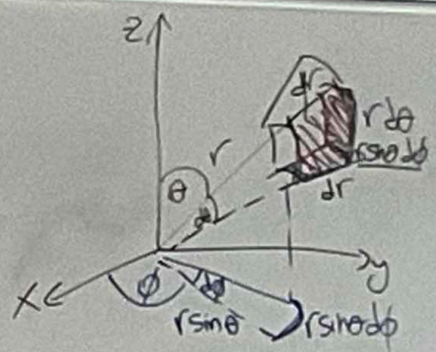

Coordinate systems
rectangular (x,y,z)
z
dy
dz
dx
x
y
dl = dy, dz, dx
dA = dydz, dxdy, dxdz
dV = dxdydz
Cylindrical - (r, θ, z)
z
r
(x,y,z)
y
θ
x
rcosθ
rsinθ
y
x
θ
(x,y,z=0)
x = rcosθ
y = rsinθ
z = z
r = √(x² + y²)
tanθ = y/x
r = distance from z-axis 0 ≤ r < ∞
θ = angle from positive x-axis 0 ≤ θ < 2π
z = height above/below xy plane
-∞ < z < ∞
1D


z
dz
rdθ
r
dr
x
θ
dθ
y
Spherical : (r, ϕ, θ)
z
(x,y,z)
rcosθ
r
θ
x
ϕ
y
rsinθ
(x,y,z=0)
x=rsinθcosϕ
y=rsinθsinϕ
z=rcosθ
dl = dr, dz, rdθ
dA = rdθdz, rdθdr, drdz
dV = rdθdrdz
r = distance from origin
ϕ = angle from positive x-axis
θ = angle from positive z-axis

$r=\sqrt{x^2+y^2+z^2}$ $0\le r<\infty$

$\tan\phi=\frac{y}{x}$ $0\le\phi<2\pi$

$\cos\theta=\frac{z}{r}$!!! $0\le\theta\le\pi$



切分布的球壳(bo)

$q_{shell}=\iint b_0\,dA$

$=b_0 r^2\sin\theta\,d\theta\,d\phi$

$=b_0R^2\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi$

$=b_0R^2\times(-\cos\pi+\cos 0)\times 2\pi=4\pi R^2 b_0$

$dl=dr,\ r\,d\theta,\ r\sin\theta\,d\phi$

$dA=r^2\sin\theta\,d\theta\,d\phi,\ r\sin\theta\,dr\,d\phi,\ r\,d\theta\,dr$

$dV=r^2\sin\theta\,dr\,d\phi\,d\theta$



在外面($r>R$) → $4\pi r^2 b_0$

3.10物理L

Electric field of continuous charge distributions

$$\vec{E}(\vec{r})=\int\frac{k\,dq(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}=\begin{cases}\int\frac{k\lambda\,dL(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} & 1D\\ \iint\frac{k\sigma\,dA(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} & 2D\\ \iiint\frac{k\rho\,dV(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} & 3D\end{cases}$$

$\vec{r}$: constant

$\vec{r}'$: variable

eg: Electric field of a uniformly charged ring



1D: $\vec{E}(r)=\int\frac{k\lambda\,dl(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

$\vec{r}=(0,0,z)=z\hat{z}$

$\vec{r}'=(R\cos\theta,R\sin\theta,0)$

$dl=R\,d\theta$

$$E_x=\int_0^{2\pi}\frac{k\lambda R\,d\theta\,(-R\cos\theta)}{(R^2+z^2)^{\frac{3}{2}}}\quad d\theta=0$$

$$E_y = \int_0^{2\pi} \frac{k\lambda R^2 \sin\theta}{(R^2+z^2)^{\frac{3}{2}}} d\theta = 0$$

$$E_z = \int_0^{2\pi} \frac{kR\lambda d\theta z}{(R^2+z^2)^{\frac{3}{2}}} = \frac{kR\lambda z}{(R^2+z^2)^{\frac{3}{2}}} \int_0^{2\pi} d\theta = \frac{k2\pi R\lambda z}{(R^2+z^2)^{\frac{3}{2}}} \hat{z}$$

Above center of a ring

Point away from the wire

Infinite line of charge $\vec{E}(r) = \frac{2k\lambda}{r}\hat{r}$



point charge $\vec{E}(\vec{r}) = \frac{kq(\vec{r}-\vec{r_0})}{|\vec{r}-\vec{r_0}|^3}$ $\vec{r_0}$: 源量 $\vec{r}$: 要求的点

eg: insulating disk of radius R charged with $\sigma(r) = \alpha r^2$ using $\int \frac{x^3 dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x^2+2a^2}{\sqrt{x^2+a^2}} + C$

$dq = \sigma dA = \alpha r^2 r d\theta dr$ $\vec{r'} = r\cos\theta\hat{x} + r\sin\theta\hat{y}$ $\vec{r} = h\hat{z}$

$\vec{r} - \vec{r'} = h\hat{z} - r\cos\theta\hat{x} - r\sin\theta\hat{y}$ $|\vec{r}-\vec{r'}| = \sqrt{h^2+r^2}$

Symmetry: $E_x = E_y$ $\vec{E} = +\hat{z}$

$$\vec{E} = \iint \left(\frac{k\alpha r^3 d\theta dr}{(\sqrt{h^2+r^2})^3} (h\hat{z} - r\cos\theta\hat{x} - r\sin\theta\hat{y}) \right)$$

$$= k\alpha h\hat{z} \int_0^{2\pi} d\theta \int_0^R \frac{r^3 dr}{(h^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi\alpha k h \left[\frac{R^2+2h^2}{(R^2+h^2)^{\frac{1}{2}}} - 2h \right] \hat{z}$$

$$E_y = \int_0^{2\pi} \frac{k\lambda R^2 \sin\theta}{(R^2+z^2)^{\frac{3}{2}}} d\theta = 0$$

$$E_z = \int_0^{2\pi} \frac{kR\lambda d\theta z}{(R^2+z^2)^{\frac{3}{2}}} = \frac{kR\lambda z}{(R^2+z^2)^{\frac{3}{2}}} \int_0^{2\pi} d\theta = \frac{k2\pi R\lambda z}{(R^2+z^2)^{\frac{3}{2}}} \hat{z}$$

Above center of a ring

Point away from the wire

Infinite line of charge $\vec{E}(r) = \frac{2k\lambda}{r}\hat{r}$



point charge $\vec{E}(\vec{r}) = \frac{kq(\vec{r}-\vec{r_0})}{|\vec{r}-\vec{r_0}|^3}$ $\vec{r_0}$: 源量 $\vec{r}$: 要求的点

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$dq = \sigma dA = \alpha r^2 r d\theta dr$ $\vec{r'} = r\cos\theta\hat{x} + r\sin\theta\hat{y}$ $\vec{r} = h\hat{z}$

$\vec{r} - \vec{r'} = h\hat{z} - r\cos\theta\hat{x} - r\sin\theta\hat{y}$ $|\vec{r}-\vec{r'}| = \sqrt{h^2+r^2}$

Symmetry: $E_x = E_y$ $\vec{E} = +\hat{z}$

$$\vec{E} = \iint \left(\frac{k\alpha r^3 d\theta dr}{(\sqrt{h^2+r^2})^3} (h\hat{z} - r\cos\theta\hat{x} - r\sin\theta\hat{y}) \right)$$

$$= k\alpha h\hat{z} \int_0^{2\pi} d\theta \int_0^R \frac{r^3 dr}{(h^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi\alpha k h \left[\frac{R^2+2h^2}{(R^2+h^2)^{\frac{1}{2}}} - 2h \right] \hat{z}$$

Since $\vec{E}$ is conservative

$$\Delta U = -W = -\int_{\vec{r}_1}^{\vec{r}_2} q_0 \vec{E} \cdot d\vec{l} = -q_0 \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{l}$$

φ: Electric potential is defined as the potential energy per unit charge $\varphi = \frac{U}{q_0}$

$$\therefore \Delta\varphi = \varphi(\vec{r}_2) - \varphi(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{l}$$

A collection of charge

$$\varphi(\vec{r}) = \sum \frac{kq}{|\vec{r} - \vec{r}_i|}$$

A continuous distribution of charge

$$d\varphi(\vec{r}) = \frac{kdq}{|\vec{r}-\vec{r}'|} \quad \varphi(\vec{r}) = \int \frac{kdq}{|\vec{r}-\vec{r}'|} = \begin{cases} 1D\ dq=\lambda dl & \varphi(\vec{r}) = \int \frac{k\lambda dl}{|\vec{r}-\vec{r}'|} \\ 2D\ dq=\sigma dA & \varphi(\vec{r}) = \iint \frac{k\sigma dA}{|\vec{r}-\vec{r}'|} \\ 3D\ dq=\rho dv & \varphi(\vec{r}) = \iiint \frac{k\rho dv}{|\vec{r}-\vec{r}'|} \end{cases}$$

Remark: 1) the equation for the potential assumes that the reference point $r_0 \to \infty$ $\varphi(r_0 \to \infty) = 0$

2) We have learned 2 ways

1) By definition $\varphi(\vec{r}) - \varphi(\vec{r_0}) = -\int_{r_0}^{r} \vec{E} \cdot d\vec{l}$

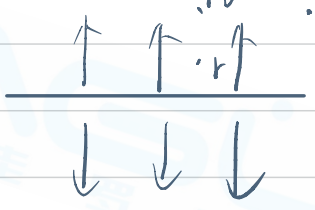
2) $\varphi(\vec{r}) = \int \frac{k\,dq}{|\vec{r}-\vec{r}'|}$ $(\vec{r_0} \to \infty)$

In general use Gauss law to find $\vec{E}$ → use (2) (symmetry)

Eg: φ of infinite line

Gauss law $E \cdot 2\pi r l = 4\pi k \lambda L$ $E = \frac{2k\lambda}{r}\hat{r}$

$\varphi(\vec{r}) - \varphi(\vec{r_0}) = -\int_{\vec{r_0}}^{\vec{r}} \vec{E}\cdot d\vec{l}$



$\varphi(\vec{r}) - \varphi(\vec{r_0}) = -\int_{\vec{r_0}}^{\vec{r}} \vec{E}\cdot d\vec{l} = -\int_{\vec{r_0}}^{\vec{r_0'}} \vec{E}\cdot d\vec{l} - \int_{r_0'}^{\vec{r}} \vec{E}\cdot d\vec{l}$

$E \perp l \to = 0$

$= -\int_{r_0'}^{\vec{r}} \frac{2k\lambda}{r}\hat{r}\cdot dr\hat{r} = -\int_{r_0}^{r} \frac{2k\lambda}{r} dr = -2k\lambda \int_{r_0}^{r} \frac{1}{r} dr$

$= -2k\lambda \ln\left(\frac{r}{r_0}\right)$

Eg2 potential of a ring (uniformly charged) (不对称、法2)

Sol: $\vec{r}=(0,0,z)$ $\vec{r}'=(R\cos\theta, R\sin\theta, 0)$ $dq=\lambda dl=\lambda R d\theta$

$|\vec{r}-\vec{r}'|=\sqrt{z^2+R^2}=\text{const}$

$\varphi(0,0,z)=\int_0^{2\pi}\frac{k\lambda R d\theta}{\sqrt{z^2+R^2}}=\frac{k\lambda R 2\pi}{\sqrt{z^2+R^2}}$ $\lambda R 2\pi=Q$ $\varphi(0,0,z)=\frac{kQ}{\sqrt{z^2+R^2}}$

Eg3 Electric potential of a uniformly charged disk

$\vec{r}=(0,0,z)$ $\vec{r}'=(r'\cos\theta, r'\sin\theta, 0)$ $dA=r'd\theta dr'$

$|\vec{r}-\vec{r}'|=\sqrt{r'^2+z^2}$ $\varphi=\int_0^R\int_0^{2\pi}\frac{k\sigma r' d\theta dr'}{\sqrt{z^2+r'^2}}=2\pi k\sigma\int_0^R\frac{r'dr'}{\sqrt{z^2+r'^2}}=2\pi k\sigma\left(\sqrt{z^2+r'^2}\Big|_{r'=0}^{r'=R}\right)$

Eg4 φ of a uniformly charged spherical shell

① outside $EA = E \times 4\pi r^2 = 4\pi k Q_{in}$ $E = \frac{kQ}{r^2}\hat{r}$

$\varphi(\vec{r}) - \varphi(\vec{r_0}) = -\int_{\vec{r_0}}^{\vec{r}} \vec{E} \cdot d\vec{l}$ like a point charge $\vec{r_0} \to \infty$ $\varphi(\vec{r_0}) = 0$.

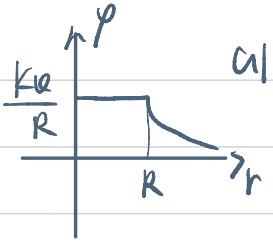
$\varphi(\vec{r}) = \frac{kQ}{r}$

② $r < R$ $EA = 4\pi k Q_{in} = 0$ $E = 0$ $\varphi(\vec{r}) - \varphi(\vec{r_0} \to \infty) = \int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{R} \vec{E}_{out} \cdot d\vec{l}$

$-\int_{R}^{r} \vec{E}_{in} \cdot d\vec{r}$ $= \frac{kQ}{R}$

0

$$\varphi(r) = \begin{cases} \frac{kQ}{R} & r \le R \quad \text{constant} \\ \frac{kQ}{r} & r > R \quad \text{like a point charge} \end{cases}$$



always continuous

$\vec{E}$ isn't

Electrostatic energy stored in a system of N point charges

What is the work required to bring all the charges from $r_0 \to \infty$ to their final position?

The first one comes for free

$q_1 \quad q_2 \quad q_3 \quad \cdots\cdots\cdots \infty$

$V_{12} = q_2 \cdot \frac{kq_1}{r_{12}}$ (potential at q_2 due to q_1)

$V_3 = q_3 \cdot \frac{kq_1}{r_{13}} + q_3 \cdot \frac{kq_2}{r_{23}}$

$$U = \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{kq_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ (i \neq j)}}^{N} \frac{kq_i q_j}{r_{ij}} = \frac{1}{2} \sum_{i=1}^{N} q_i \underbrace{\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{kq_j}{r_{ij}}}_{= \varphi_i}$$

potential due to all the charges (except q_i) at the position of q_i

Energy stored in a continuous charge distribution



$U = \frac{1}{2} \int dq\, \varphi$ φ is the potential at the position of dq

$$U = \frac{1}{2} \int dq\, \varphi = \frac{\varepsilon_0}{2} \iiint E^2 dv$$

over the charge body

over all space

Eg 正方体边长a 八个角 (-e) 中间(2e)

$$U=\sum_{j}^{N}\sum_{i}^{N}\frac{kq_iq_j}{r_{ij}}=12\frac{k(-e)(-e)}{a}+12\frac{k(-e)(-e)}{\sqrt{2}a}+4\frac{k(-e)(-e)}{\sqrt{3}a}$$

$$+8\frac{(-e)(2e)}{\frac{\sqrt{3}a}{2}}$$

$U>0$ We need to invest work

$U<0$ the system wants to stay together

Eg: ··· in a spherical thin shell total Q. uniform charge

$$U = \frac{1}{2}\int dq\,\varphi(r=R) = \frac{1}{2}\int dq\,\frac{kQ}{R} = \frac{1}{2}\frac{kQ}{R}\int dq = \frac{1}{2}\frac{kQ^2}{R}$$

Sewnd approach $U = \frac{\varepsilon_0}{2}\left[\int_{r<R}^{R} 0^2\,dv + \int_{R}^{\infty}\left(\frac{kQ}{r^2}\right)^2 4\pi r^2\,dr\right]$ $k = \frac{1}{4\pi\varepsilon_0}$

$$= \frac{kQ^2}{2}\left[-\frac{1}{r}\Big|_R^{\infty}\right] = \frac{kQ^2}{2R}$$

Equipotential surfaces = surfaces connecting points having the same value of electric potential

of between any equipotential surface and its neighbour has the same

value of ϕ $\Rightarrow$ more dense equipotential surfaces $\Rightarrow$ stronger E

E is always perpendicular to equipotential surfaces.

Calculating the field from the potential

$$\varphi(\vec{r}) - \varphi(\vec{r_0}) = -\int_{\vec{r_0}}^{\vec{r}} \vec{E} \cdot d\vec{l} \quad \Rightarrow \quad E(x) = \frac{-d\varphi}{dx} \quad \Rightarrow E = -\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right) = -\vec{\nabla}\varphi$$

eg: $\varphi(x,y,z) = x^3y + z \qquad Ex = -3yx^2 \quad Ey = -x^3 \quad Ez = -1$

$|\vec{\nabla}\varphi|$: magnitude of the maximum slope

$\vec{E} = -\vec{\nabla}\varphi$: $\vec{E}$ points in the direction in which φ decreases more quickly

Electric fields and electric potential around conductors

Conductor: charge can flow inside the conductor.

Electrostatic equilibrium state: charge doesn't move anymore

For an idea conductor in an electrostatic equilibrium state

Rule ① $\vec{E}=0$ inside the material (because if $\vec{E}\neq 0$, the charge will move)

② $\rho=0$ inside the conductor ($q_{in}=0$)

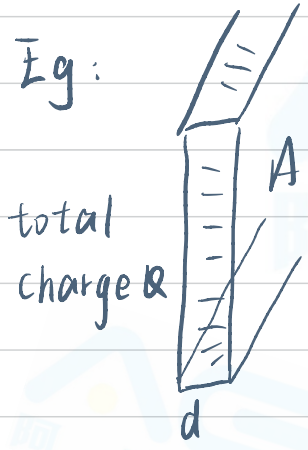
③ V = constant in the conductor

④ very near the surface on the outer side $\vec{E}=4\pi k\sigma\hat{n}$

Conductor with a cavity inside → Faraday's cage



Eg:



Find ① charge distribution ② $\vec{E}$ everywhere

Reminder: infinite plane 无限常数

